Coordinating Numerical and Linear Units: Elementary Students' Strategies for Locating Whole Numbers on the Number Line[1]

Geoffrey B. Saxe
University of California, Berkeley

Meghan M. Shaughnessy
University of Michigan

Maryl Gearhart, Lina C. Haldar
University of California, Berkeley

Abstract

Two investigations of 5th graders' strategies for locating whole numbers on number lines revealed patterns in students' coordination of numerical and linear units. In Study 1, we investigated the role of a measurement context (race course) in students' placements of 3 numbers on an empty number line. For one group (n=24), the line was presented as a 'race course,' and for a second group (n=24), the line was presented as a conventional number line. Most students in both groups placed *consecutive* whole numbers at appropriate linear distances, but the race course group was more likely to place *non-consecutive* whole numbers at appropriate linear distances. In Study 2 (n=24), students placed numbers on lines marked with two numbers. Most students placed a third number appropriately when the marked numbers were *consecutive* whole numbers, but not when the labeled numbers were non-consecutive whole numbers (differing by a numerical value of 2 or 3 instead of 1).

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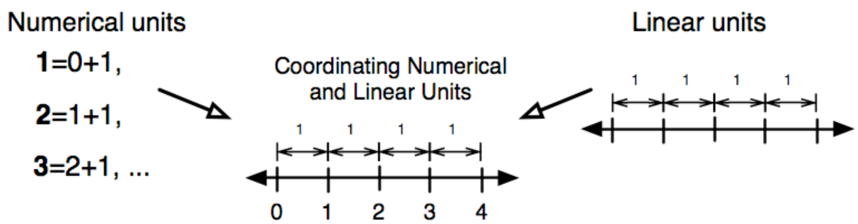
The number line is a geometric interpretation of number, a representation of a straight line subdivided into numbered linear units. Representation of numbers on the line is guided by mathematical conventions such as tick marks to identify points, and arrows to indicate the line extends indefinitely in both directions. Interpretation of number lines requires the construction and coordination of the ideas of numerical unit and linear unit (Figure 1).

Figure 1. The number line as a hybrid representation involving a coordination of linear and numerical units.



The studies reported here are steps toward building a knowledge base to inform instructional uses of number lines. We produced (a) qualitative analyses of 5th grade students' approaches to coordinating numeric and linear units on number lines, as well as (b) quantitative comparisons of students' approaches to unitizing when lines were presented as conventional number lines and race courses. The purpose of Study 1 was to determine *how students construct and use linear units to place numbers on open number lines* presented as either 'race courses' or conventional lines. The purpose of Study 2 was to investigate students' *interpretation and use of linear unit when two numbers are marked on a number line.*

Study 1: Students' Construction of Linear Units on Open Number Lines in Thematic and Number Line Contexts

Students positioned three numbers on an open line. The 'measurement-explicit' students placed cartoon animals on a cartoon race course line, and the 'measurement-implicit' students placed numbers on a blank number line.

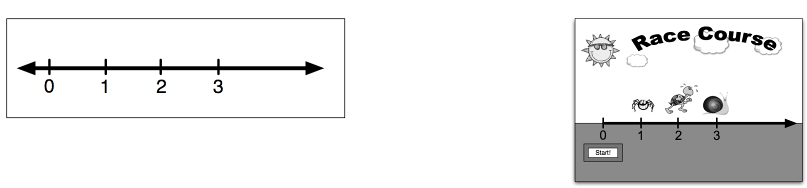
Method

Forty-eight fifth grade participants were drawn from schools in urban districts. Students were interviewed individually, and all interviews were videotaped.

Students were introduced to either the number line or the race course context.

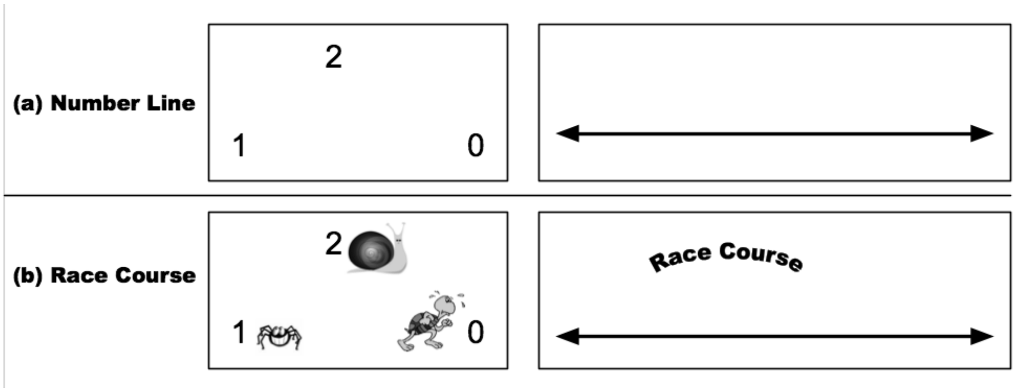
Figure 2. Examples of the number line and race course tasks.

(a) Number Line (b) Race Course



Students were then asked to place three numbers on lines (or race courses) in a series of tasks (Figure 3). Block I consisted of *consecutive numbers tasks*: (a) 0, 1, 2, and (b) 5, 6, and 7. Block II consisted of tasks in which either only two of the three whole numbers were consecutive (9, 10, 13 and 9, 12, 13), or none of the three whole numbers was consecutive (7, 11, 14).

Figure 3. Examples of the Block I number line and race course tasks.

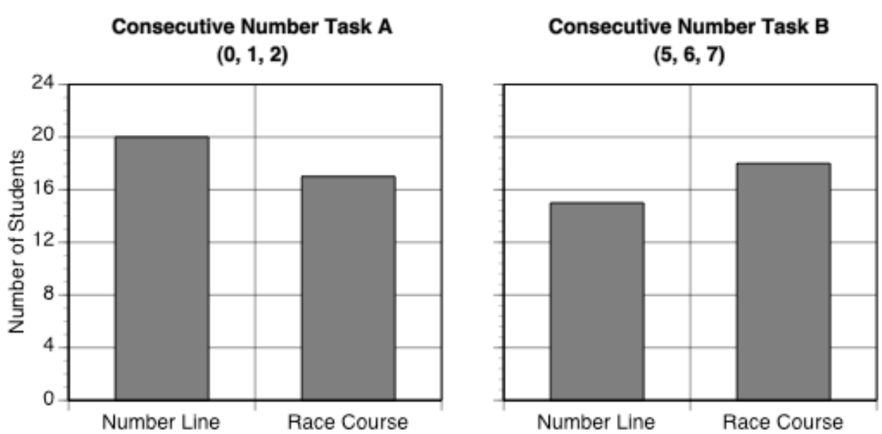


Students' responses were coded with two schemes. Strategy codes described students' coordination of numerical and linear units: *No Unit* -- no evidence that the student constructed this coordination (e.g., students placed the values in order and approximately equidistant regardless of the values of the target numbers; *Qualitative* strategy -- evidence that students conceptualized the number series in terms of relative magnitudes (e.g., 'There are more numbers in here (gesturing to the line between one pair of numbers) so I put it over here.'); *Unit* strategy -- when students constructed and used a linear unit to place additional numbers on the line, and the relative distances between numbers were appropriate. Tool codes described the ways that students improvised a physical resource or gestural resource to mediate their placements of numbers on the line: *No Tool* -- no evidence of tool use; *Non-metric Tool* -- gestural activity with no apparent attention to unit interval distance; *Metric Tool* -- use was a rigid object to iterate intervals and locate points along the line. Schemes were applied by two raters, with agreement of 90% or greater; disagreements were resolved.

Results

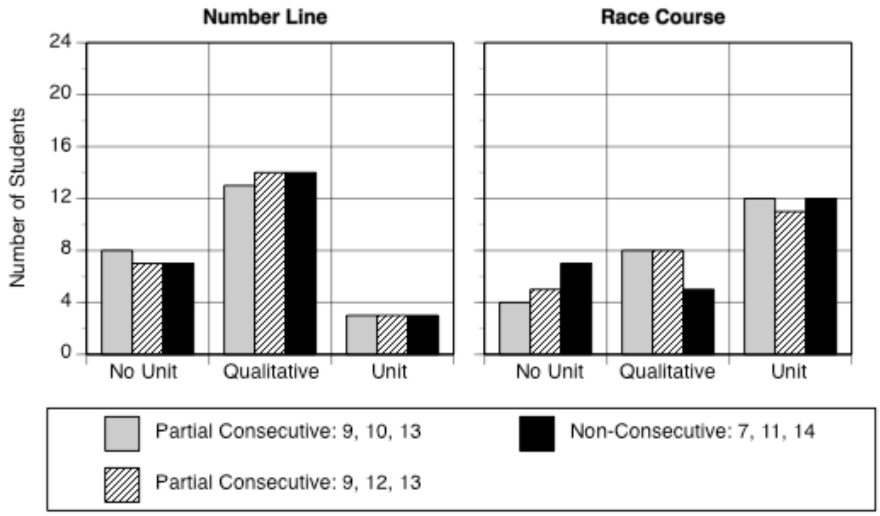
We used a +/- 20% margin of error to code students' placements on Block I consecutive number tasks as 'correct' or 'incorrect.' The majority of students in each group responded correctly to both tasks (Figure 4).

Figure 4. Number of students in the number line and race course groups coded as correct on the consecutive number tasks A (0, 1, 2) and B (5, 6, 7).



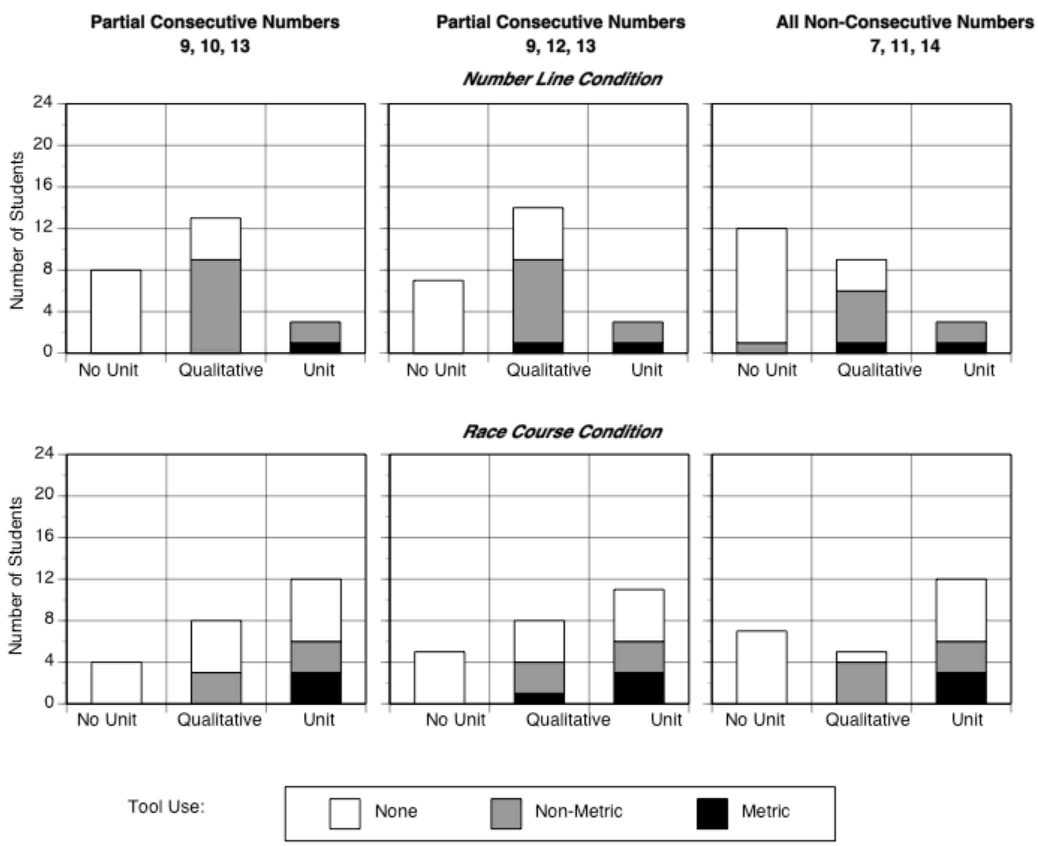
To analyze students' performance on the Block II partial consecutive and all-nonconsecutive numbers tasks, we summed the use of Unit strategies across the three tasks, with scores ranging from 0 to 3. A Mann-Whitney U-Test comparing groups revealed a main effect for task context, U (N=48)= 181.500, p = .006 (two-tailed). Students in the Race Course group used Unit strategies more frequently (*Mdn* = 1, Mean Rank=28.94) than students in the Number line group (*Mdn* = 0, Mean Rank=20.06) (Figure 5).

Figure 5. Strategy types as a function of number line and race course contexts and tasks.



Students whose strategies were coded No Unit rarely used tools; students who used Qualitative or Unit strategies varied in their use of tools (Figure 6).

Figure 6. Strategy types by task and by context



Study 2: Using a Marked Interval to Identify Additional Points on A Number Line

The purpose of Study 2 was to investigate how students used a linear interval—two marked integers on a number line—to position a third number.

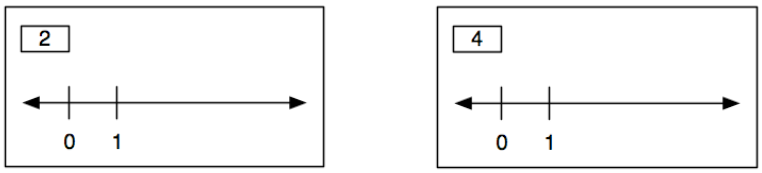
Method

Twenty-four fifth grade participants were drawn from the same elementary school in Study 1. Participants were interviewed individually.

In Block I tasks, the unit of (0,1) was defined on the number line, and the student was required to place a mark for a third number on the line (Figure 7).

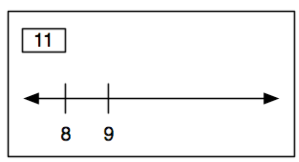
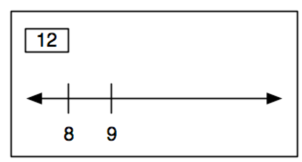
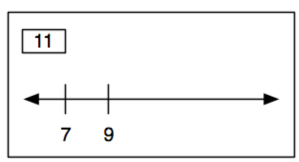
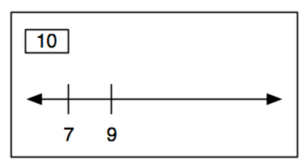
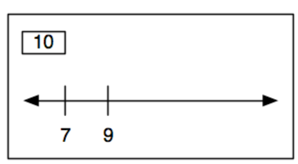
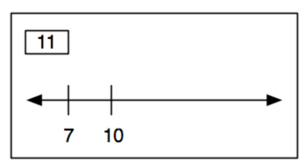
Figure 7. Block I task cards with marked (0,1) interval.

(a) Single Unit Target (b) Multiunit Target



Block II consisted of 6 tasks that provided an interval other than 0, 1 marked on the number line, and the student was asked to place a third target number (Figure 8).

Figure 8. Block II tasks.

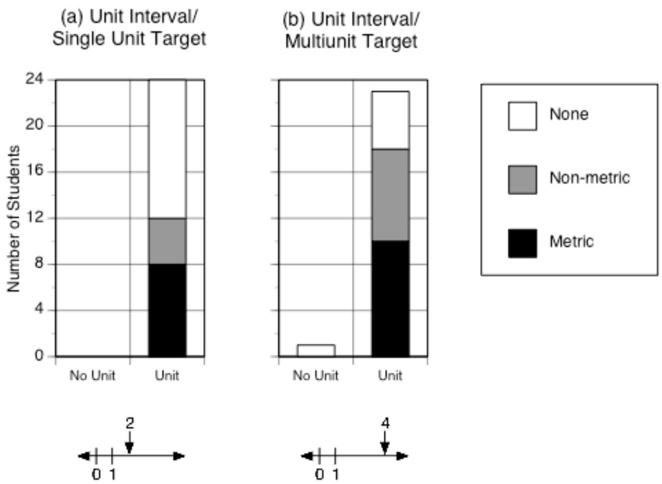
	(a)	(b)
(i) Unit Interval: Multiunit Target		
(ii) Multiunit Interval: Multiunit Target (replication)		
(iii) Multiunit Interval: Fractional Target		

Strategy use codes described students' coordination of numerical and linear units. *No Unit* was coded when there was no evidence that the student constructed this coordination. *Unit* strategy was coded when the student iterated and or partitioned the marked interval in order to place the target number at an approximately appropriate distance from the marked interval. Tool use codes described the ways that students appropriated some kind of physical tool to mediate their placements of numbers on the line. *No Tool* was coded when there was no evidence that the student was using any kind of tool. A *Non-metric Tool* was gestural activity, such as pulsing gestures along the line, with no apparent attention to unit interval distance. A *Metric Tool* use was an informal tool, such as a fingertip or finger joint or a marker cap, used to iterate intervals and locate points along the line. Schemes were applied by two raters, with agreement of 90% or greater; disagreements were resolved.

Results

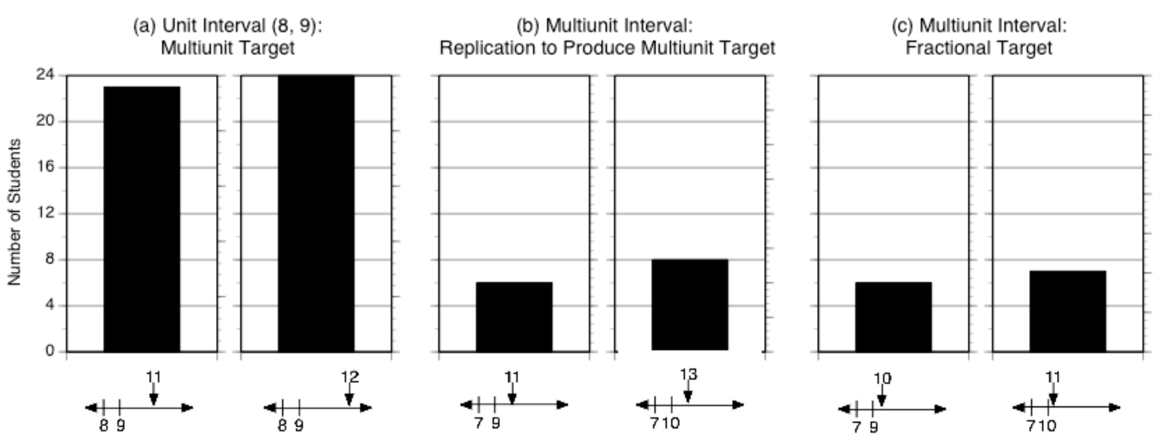
On *Block I* tasks, almost all students used an appropriate Unit strategy in order to locate the target number. Students' strategies were not related to their use of tools (Figure 9).

Figure 9. Relationship between strategy and tool use on Block I tasks with (0,1) marked.



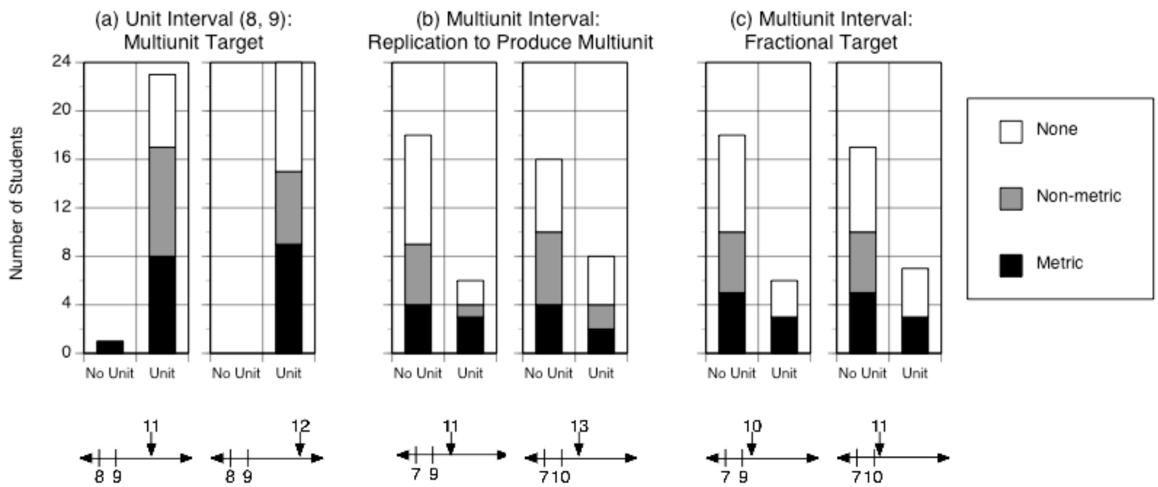
The results for *Block II tasks with marked intervals other than (0,1)* are shown in Figure 10. We assigned 1 point if the student used a Unit strategy to locate the target number on the line. There were two tasks for each type, so scores ranged from 0 to 2 for each item type. Friedman's ANOVA revealed a statistically significant effect of item type on use of unit strategy (2,, n = 24) = 33.552*,* p < .001. Follow-up pair-wise comparisons with Wilcoxon Signed Ranks Test indicated that students were more likely to use an appropriate unit strategy on the *unit interval (8,9) with multiunit target* tasks (Mdn = 2.0, Mean Rank = 2.75) than on either the *multiunit interval with a multiunit target* tasks (*Mdn* = 0, Mean Rank = 1.65), *Z*(N=24)= -4.001, *p* < .001 (two-tailed), or the *multiunit interval with a fractional target* tasks (*Mdn* = 0, Mean Rank = 1.60), *Z* (N=24)= -4.066, *p* < .001 (two-tailed). There was no significant difference in performance on either of the multiunit interval task types, *Z* (N=24)= -.577 *p* =.564 (two-tailed).

Figure 10. Number of students using a Unit strategy on each Block II task.



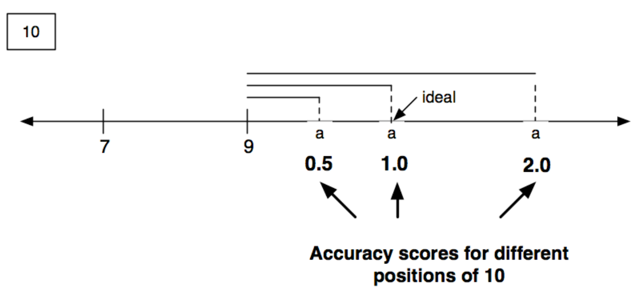
There was no relationship between strategy and tool use in the Block II tasks (Figure 11).

Figure 11. Variation in students' tool use by strategy type.



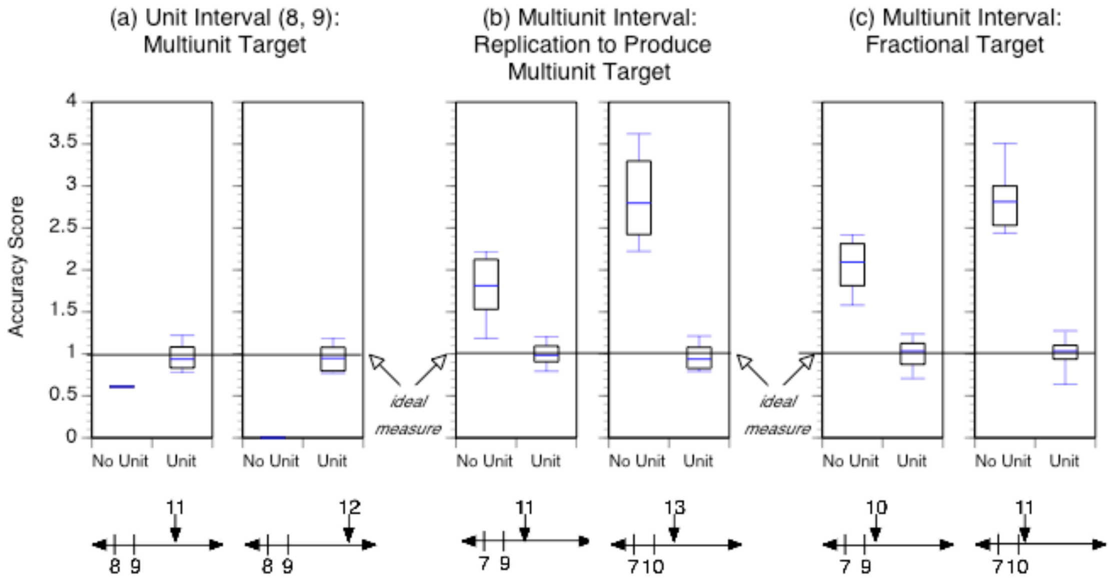
We computed accuracy scores for each placement of a number on the six Block II tasks (see examples of three different solutions each marked "a" in Figure 12).

Figure 12. Computation of accuracy scores for placements of numbers on the line.



The box plots (Figure 13) reveal a pattern of students' methods of coordinating numerical and linear units. For the multiunit interval → multiunit target tasks ((7, 9: *11*) and (7, 10: *13*)), and multiunit interval → fractional target tasks ((7, 9: *10*) and (7, 9: *11*)), students' placements tended to be overestimates of the precise position by factors of 2 and 3. Students tended to iterate the given multiunit interval as a linear unit to locate the target number, and, in doing so, they confused a multiunit interval with a unit interval.

Figure 13. Boxplots representing accuracy of placements



Discussion

Our findings indicate that many fifth graders do not have a rich and generative understanding of the coordinated relation of numerical and linear units on the number line. Opportunity to learn is a likely factor – number lines are not emphasized in elementary curriculum, and the number lines that do appear tend to be canonical lines with equally spaced tick marks representing a fixed numerical progression. A key feature of our task design was the contrast between canonical and non-canonical number lines, and our findings indicated that students' interpretations and constructions of canonical lines can inflate estimates of students' generative understanding of number line principles and conventions. We believe that our tasks have promise for elementary curriculum and instruction.

A second pattern in our findings was the *role of a thematic context in supporting student' insights about number lines.* While representational contexts may introduce confusing mathematical ideas, our findings indicate that thematic contexts may sometimes provide students a foothold into mathematical ideas that are difficult to communicate. The idea of a race on a linear path where linear distance is a core component of what it means to 'race' appeared to cue students' recognition that linear and numeric units must be coordinated to determine 'how far' someone has raced.

The third pattern is the *independence of tool use and strategy use.* The absence of an association between tool use and strategy makes clear that what develops is students' understandings of the coordination of linear and numeric units.

In our current project, *Learning Mathematics through Representations,* we are building on these findings as we develop and investigate new curriculum and instructional methods using the number line as the principle representational context.