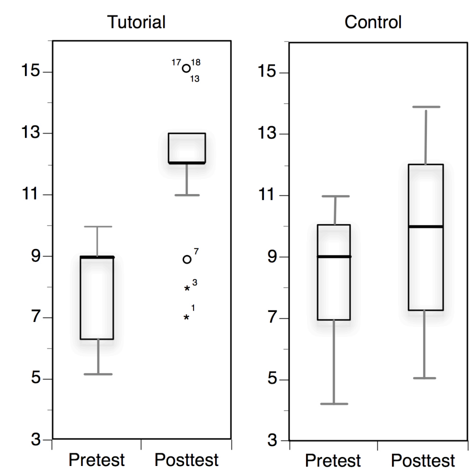
Tutorial Study: Supporting Generative Thinking about the Integer Number Line
Learning Mathematics through Representations
UC Berkeley
(poster brief)

The tutorial study poster reports evidence of the influence of a tutorial 'communication game' on fifth graders' understanding of number line principles for integers. Students matched for classroom and pretest score were randomly assigned to a tutorial (n=19) and control group (n=19).

The tutorial group students played a game of 13 problem types in which student and tutor each marked the same integer on a number line but could not see one another's activities. To resolve discrepant solutions, tutor and student constructed agreements about number line principles and conventions to support coordinated placements related to the principles of order, unit interval, symmetry, and absolute value. In the early problems, student and tutor used Cuisenaire rods to represent a linear distance on open number lines with only one number specified on the line; for example, with only 0 represented, the student and tutor marked the distance of 6 red rods using purple rods. In later problems, the number line became a self-referential representation when two numbers were specified; for example, student and tutor marked the position of 9 on a line with only 6 and 8 specified without using rods. The final problem engaged tutor and student in applying number line principles to the construction of negative numbers on the number line. If the first iteration of a problem type was not passed by a student, a second and then a third iteration of the same problem type was used (see Figure).



The tutorial group improved over pre- to posttest ($t(18)=7.92$, $p<.0001$) whereas the controls did not. Further, the effect size for the tutorial group was large (1.8 standard deviations). To investigate students' use of the agreements (number line principles), we produced an index of appropriate and inappropriate agreement use during play. We found that learning gains were predicted by appropriate agreement use. Students who used agreements more appropriately were more likely to have higher gain scores (r (N=19)=.56, p=.008), and students who used agreements more inappropriately were likely to have lower gain scores (r (N=19)=-.67, p=.001). To illustrate the learning trajectories of students as they worked through the tutorial problems, we will present case patterns of learning. Case examples will include patterns of pass-fail on the tutorial problems as well as microanalysis of the evolution of student solutions as they progressed through selected problems.

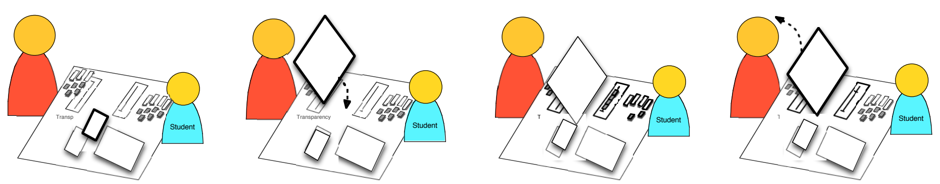
Game: Repairing and Anticipating Breaches in Communication by Constructing Mathematical Agreements

Each of the 13 coordination problem types with which students and tutors engaged required the student and tutor to:

- mark the same location on identical number lines without seeing the other's placement
- create an agreement or apply prior agreements to coordinate their interpretations of appropriate placement

Tutorial overview. Each of the two tutorial sessions lasted about 45 minutes and was videotaped. For each session, a standard procedure was used as tutor and student engaged with successive problems. The multi-phased procedure is depicted in Figure a-d.

In the problem card phase (Figure a) the tutor presented the student with a card from the card deck, a number line keyed to the problem, and rods if required by the problem. In the screen down phase (Figure b), the occluding screen was positioned at a slight oblique angle so that the tutor could surreptitiously peer over the screen to see the child's workspace. In the number line construction phase, the student and tutor constructed number lines to address the problem card (Figure c). In the screen up, positions match? phase, the screen was removed once the constructions were completed (Figure d), and the tutor asked the student to overlay the two number lines, compare point placements, and explain his/her solution. During play, a video camera was positioned to record the child's activity and interactions with the tutor.



a. problem card b. screen down c. number line construction d. screen up, positions match?

Students had three opportunities (three iterations of a problem type) to achieve coordinated point placement. The tutor proceeded differently as a function of whether or not the tutor and students successfully matched point placements.

When the points did not match, the tutor supported joint reflection on sources of the discrepancy and ways of coordinating action on subsequent problems by following several heuristics, as appropriate:

1. Refer back to agreements. The tutor asked the student if he/she considered any of the agreements when placing the point and, if so, how. Then, drawing upon the prior agreements to justify the placement, the tutor explained the placement of his/her own point (e.g., "I did mine this way because our agreements say …").

2. Refer back to problem. Sometimes a student's inappropriate construction could be interpreted as responding to a problem that was different from the one printed on the card. In such cases, the tutor oriented the child to the problem card stating, "I did mine this way because the card said to find ...."
3. Anticipate the next problem. After drawing upon the agreements or referring to the problem card, the tutor asked the student what each could do differently on the next problem card to try to place points at the same location. If players reached the third iteration of a problem type and continued to have discrepant point placements, the tutor moved on to the next problem set. The student was unaware of the distinction between types of problems.

Linear Units: The Changing Function of Cuisenaire Rods in Number Line Representations across Tutorial Problems

In the early coordination problems, adult and student initially marked places on a line as lengths of Cuisinaire Rods (Problem 2, below), or multi-unit lengths (Problem 3, below). Over time, Rods shifted from objects to be recorded on an unmarked line (Problem 2) to objects to partition the line (Problem 5), to objects to reason about linear units on the line (Problem 8). Throughout the game, student and adult negotiated agreements like those illustrated in the table below.

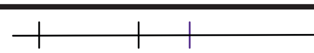
Coordination Problem	Example Solution	Agreement (if applicable)
2. Mark where 3 reds is.		Units: The distance between counting numbers must be the same.
3. Mark where 6 reds is using purple rods.		Multiunits: The distance between skip count numbers must be the same.
5. Mark where 5 reds is using red and purple rods.		
8. Mark 8. (No rods)		

Illustration of Learning Trajectory of One Student on Positive Integer Problems
One student's profile in shift from rods to self-referential number line units

Like the controls, tutorial students generally performed poorly on self-referential number line problems on the pretest. But unlike the controls, they generally passed these problems on the posttest. The case study student's trajectory was the same as the majority of the tutorial students.

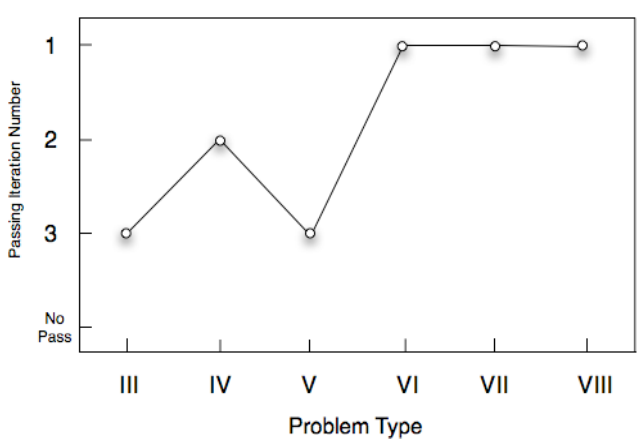
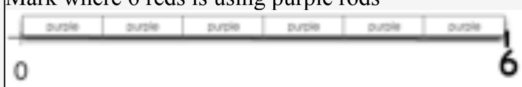
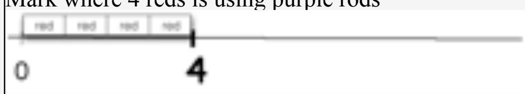
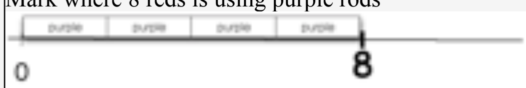
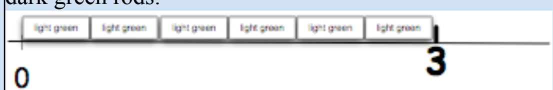
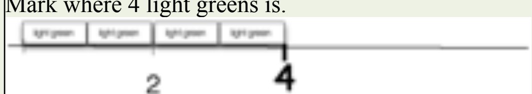
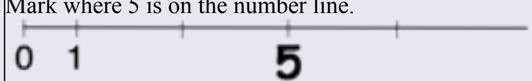
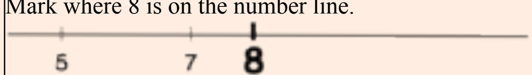


Table showing the student's trajectory from rod units to number line units (Problem Type III to Problem Type VIII, including non-passing iterations).

Problem iteration	Pass?	Problem and Student's solution	Agreement Usage
III.1	No pass	Mark where 6 reds is using purple rods 	Tutor introduced the "skip counting principle"
III.2	No pass	Mark where 4 reds is using purple rods 	Tutor brought up the "skip counting principle" again
III.3	**Pass**	Mark where 8 reds is using purple rods 	
IV.1	No Pass	Mark where 3 light greens is using both light green and dark green rods. 	Tutor introduced the "Every number has a place principle."
IV.2	**Pass**	Mark where 5 light greens is using both light green and dark green rods. 	The "Every number has a place principle" is brought up again.
V.1	No pass	Mark where 5 reds is. 	Review the "Unit distance principle" in the tutorial
V.2	No pass	Mark where 6 reds is. 	The tutor suggested it might be helpful to label other numbers on the number line
V.3	**Pass**	Mark where 4 reds is. 	--
VI.1	**Pass**	Mark where 4 light greens is. 	--
VII.1	**Pass**	Mark where 5 is on the number line. 	--
VIII.1	**Pass**	Mark where 8 is on the number line. 	--

Concluding Remarks

When students are presented with number line representations in school, they typically do not explore the generative principles underlying the geometric and numeric properties of the line, and the varied normative conventions for interpreting number line representations. The consequence for students is that many leave the elementary grades with shallow understandings that are not generative across number line representations and problems; indeed, few students in this study showed initial evidence of understanding fundamental principles of integers on the line, principles such as order, linear unit, and absolute value. The lack of focus on generative understanding of the line is unfortunate, especially in light of the frequent use of the number line in secondary mathematics and beyond. Our findings—the efficacy of the tutorial, students' learning trajectories, mediating effects of agreement use, and the construction of representations—provided empirical corroboration for the utility of the approach for supporting student learning as well as illuminating the conceptual and interactional processes that support learning trajectories for hard-to-learn, hard-to-teach ideas.

The framework and findings from this study are a core resource for our current research and development work in the Learning Mathematics through Representations project. Building on the tutorial study and other research, we are designing a curriculum unit on integers and fractions, using the number line as the principal representational context (e.g., Saxe, Gearhart, et al., 2009; Saxe, Shaughnessy, Gearhart, et al., 2009; Saxe, Shaughnessy, Shannon, et al., 2007). As we scale up to the level of the classroom, we are adapting the sequence of tutorial problem sets as a "problems of the day" lesson format that engages students with challenges requiring the class, over time, to construct and apply "Number Line Principles and Definitions." Our pilot studies are yielding very promising evidence that the approach can support students' rich understandings of integers and fractions on the number line. We believe that the principles-based approach to mathematics teaching and learning will eventually have utility in other domains and at other grade levels.