Learning Mathematics through Representations (LMR): Classroom Studies

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The Learning Mathematics through Representations (LMR) project is developing a research-based curriculum unit on integers and fractions using the number line as the principal representational context. The work builds upon Davydov's inquiries in the Soviet Union (Davydov & Tsvetkovich, 1991) and Itakura's work in science education in Japan (see Inagaki, Hatano, & Morita, 1998).

Findings from interview and tutorial studies revealed that elementary students' understandings and uses of the core principles of number lines are often neither robust nor flexible. However, from the tutorial studies we learned that students can build insight when they are challenged to construct and use explicit principles for the representation of rational number on the line; examples of core principles include order, unit interval (and the related ideas of subunit and multiunit), 0 as origin, symmetry, and absolute value. Students also build insight about the necessity of equal unit intervals when they work with number lines representing everyday measurement contexts. For example, our interview study findings indicate that 'running on a race course' can support student understanding of equivalent subunit intervals, just as 'fair-sharing' can support students' understandings of part-whole relations in area models of fractions.

The purpose of the classroom studies is the iterative development and refinement of lessons on integers and fractions. We engage in:

- ***analysis of lesson implementation*** *from classroom observations and debriefs with collaborating teachers;*
- ***analysis of student learning*** *from (a) shifts from opening to closing problems in each lesson, and (b)pre-post unit assessments;*
- ***consultation with experts,*** including a mathematician (Hyman Bass) and an educator and researcher (Deborah Loewenberg Ball)

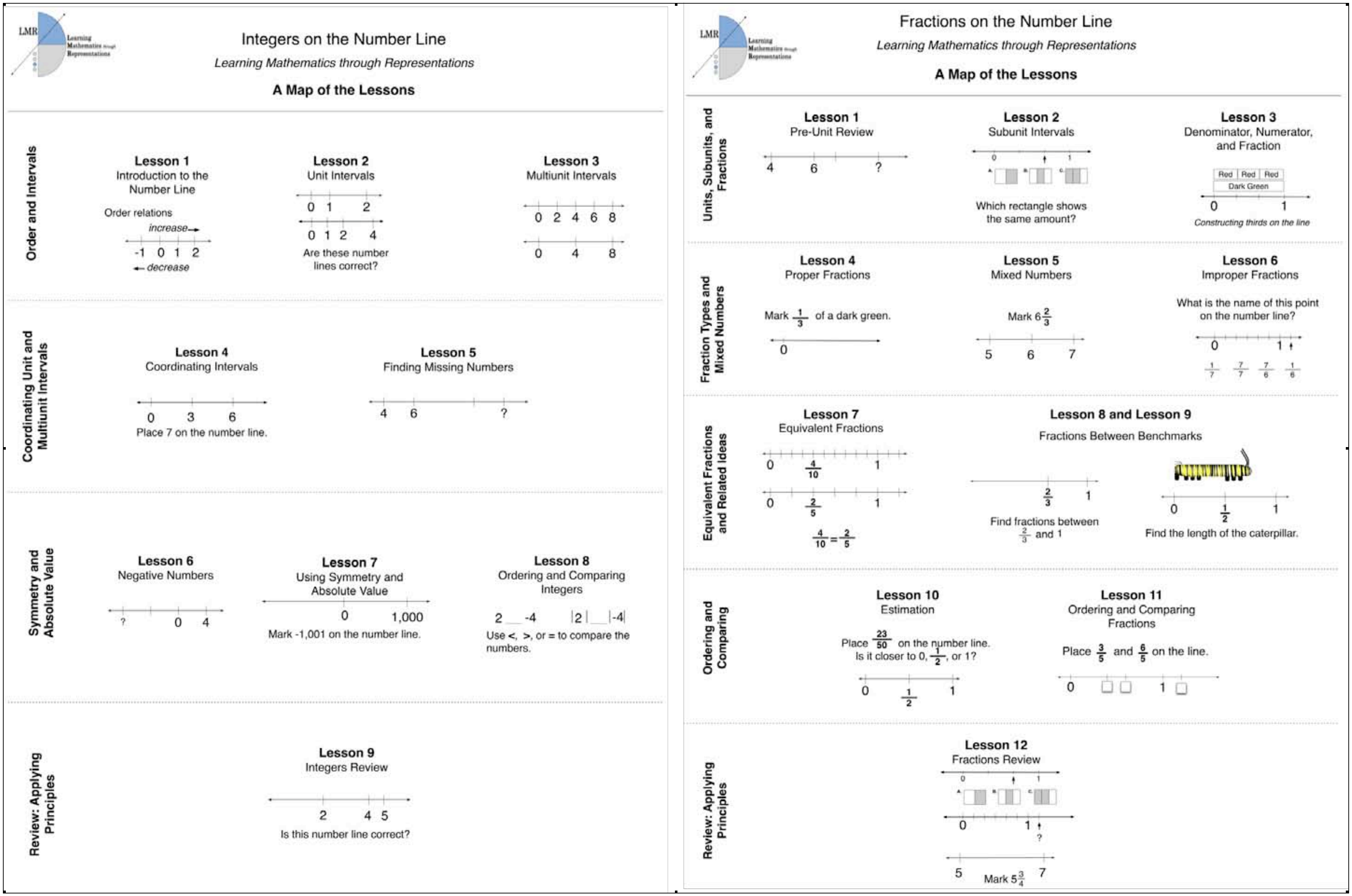
Through the various phases of our classroom studies, we have developed a 20 lesson sequence (Figure 1). When teaching the lessons, teachers engage students with the construction and flexible use of core principles of number lines as these apply to integers and fractions. Tasks and pedagogy are designed to elicit and build on student thinking in three ways:

- Tasks incorporate representations that challenge students' understandings of number line principles. For example, students are asked to identify points on lines that are not partitioned equally, locate numbers on a line that is labeled with only two numbers, or identify the principles violated by incorrect lines. Cuisenaire™ rods are critical resources to support students' developing understandings of linear units, multi-units (multiples of a linear unit), and subunits (fractional parts of a linear unit), and their uses are informed by findings on the role of rods in tutorial learning sequences. Rods are used as tools for the construction of units on the number line, location of points, and interpretation of unlabeled points.

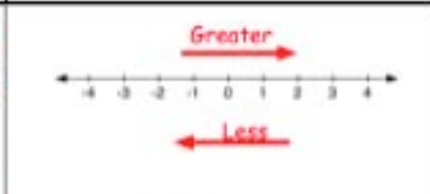
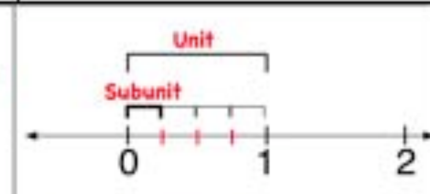
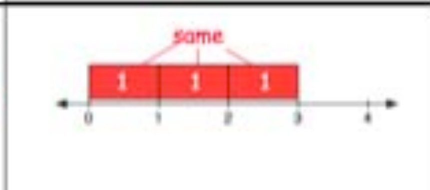
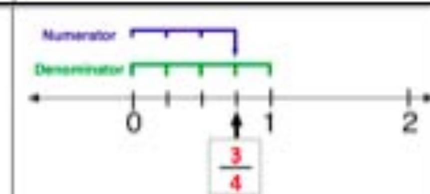
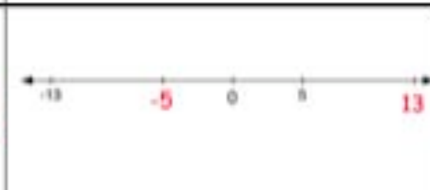
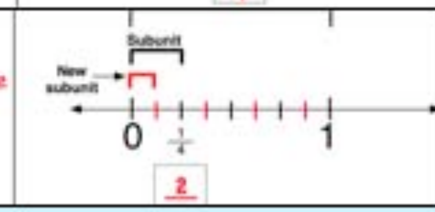
- Task formats set up contradictions for students to resolve through the construction and application of principles and definitions. For example, a multiple choice format requires students to select and defend the answer that represents their interpretation of a problem (Inagaki, Hato, & Morita, 1998); answer choices represent common responses that we identified in our interview and tutorial studies.
- Lesson formats engage teacher and students in analysis of the diversity of student ideas. A typical lesson includes: a pre-assessment (which becomes the lesson focus), opening discussion to surface student thinking, partner work on tasks that build on the pre-assessment, closing discussion to integrate students' diverse ideas, and a post-assessment that parallels the pre-assessment.

The scope and sequence of our Integers and Fractions units is shown in Figure 1 on the next page.

Figure 1. Scope and sequence of the Learning Mathematics through Representations curriculum



Principles and definitions are the core ideas of the curriculum. Throughout the lessons, students and teachers jointly construct their understanding of the principles and definitions, and students then use these ideas as resources to solve problems and communicate their mathematical reasoning. Examples are shown below.

Integers Examples			Fractions Examples		
Principle Name	Definition	Example	Principle Name	Definition	Example
Order	Numbers increase in value from left to right. Numbers decrease from right to left. Numbers at the same place have same value.		Subunit	Dividing a unit interval into equal distances creates subunits.	
Unit Interval	A unit interval is the distance from 0 to 1 or any distance of 1.		Fraction	numerator / denominator	
Symmetry	For every positive number, there is a negative number that is the same distance from 0.		Equivalent Fractions	Fractions that are in the same place but with different subunits.	

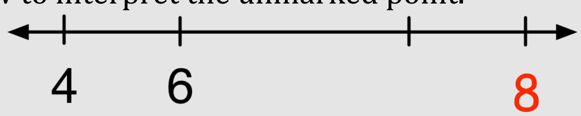
Sample Lesson

Integers Lesson 5 provides an illustration of LMR lesson design and pedagogy. Following a brief description of this lesson, we present evidence of student learning from analyses of classroom gains on integers and fractions unit assessments

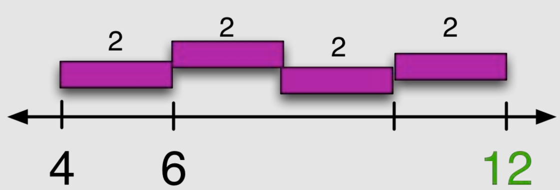
About Lesson 5

About the math. A fundamental property of the number line is that, once two numbers are labeled, the positions of all numbers are fixed, whether the line is evenly partitioned and whether its origin (0) is marked. On the line below, with 4 and 6 labeled, we can find the missing value of 12 or the value of any other point. If only 4 were labeled, all values to the right or left of 4 would be undefined.

About student understanding. Students' ideas about non-standard number lines reveal their partial understandings of unit interval. If students focus just on counting tickmarks, for example, they may label the unmarked point as 8 by counting by 1s from 6. If students are used to counting from 0, they may be confused about how to interpret the unmarked point.



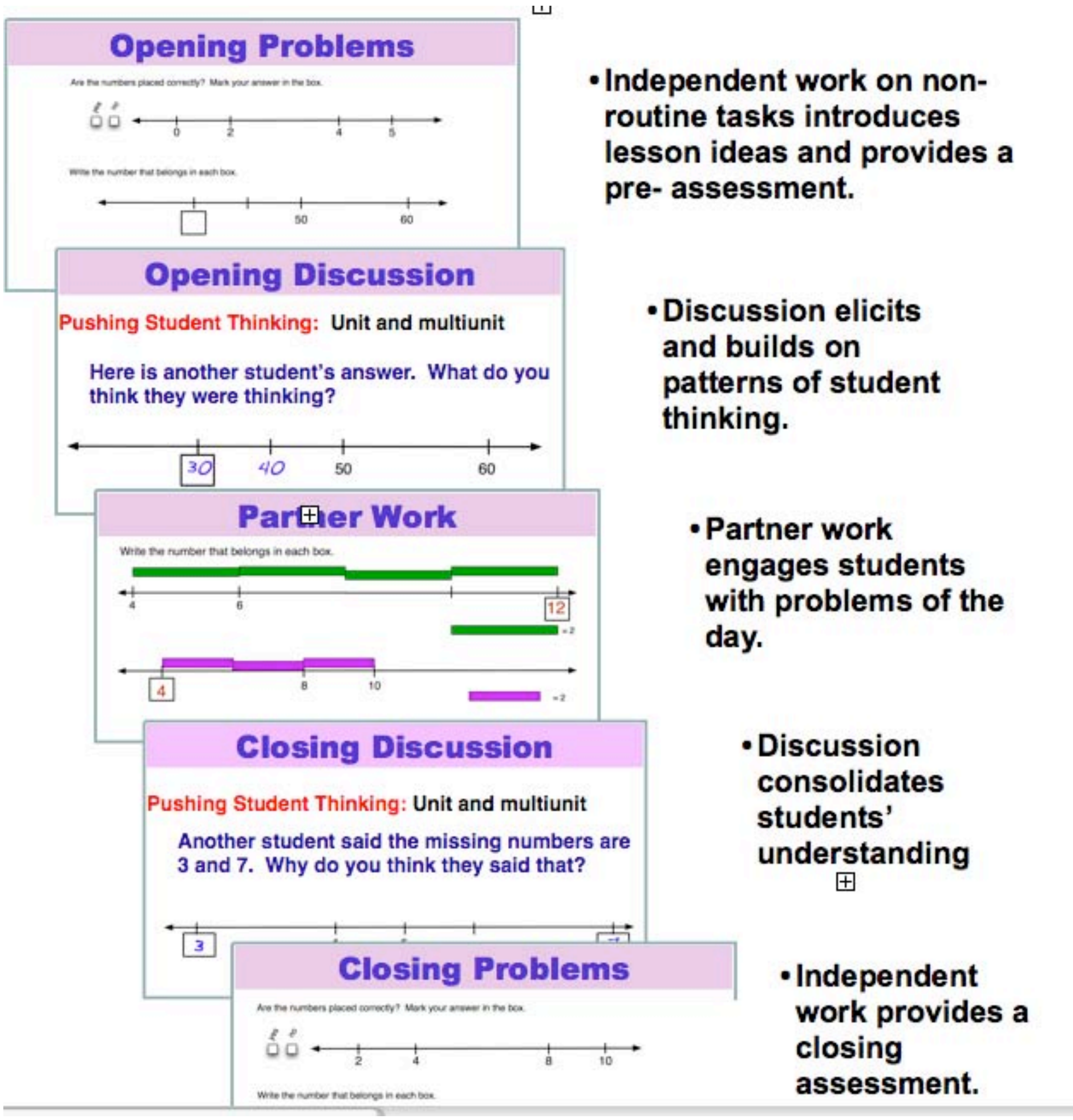
About the pedagogy. Like Lesson 4, students ask, "What's the information given?" when interpreting number lines. In Lesson 5, the marked intervals on the line are unevenly partitioned, and the leftmost integer on the line is not 0. These number lines challenge students to construct and use C-rod and number line units to solve problems. To identify the unmarked point on the sample line below, for example, students use their knowledge of C-rod relationships to measure the length of the marked interval and create a multiunit measuring tool of 2.



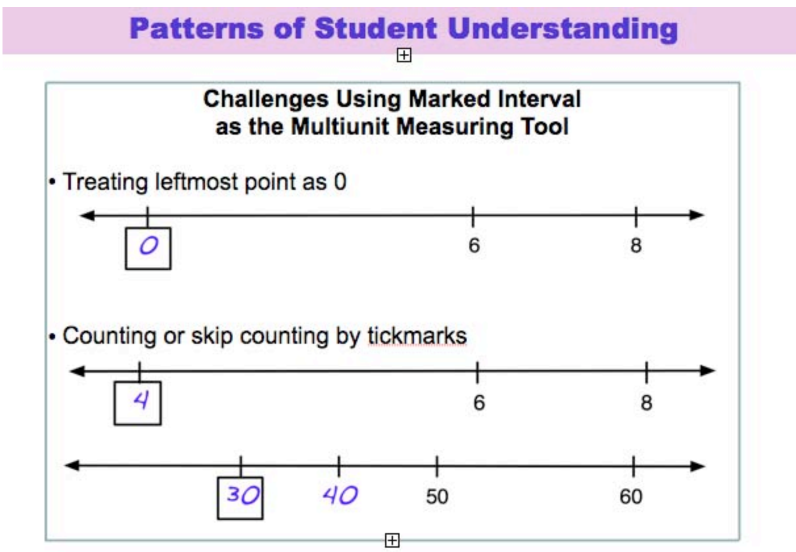
During Lesson 5, students solve more challenging problems. Some tasks require students to figure out the length of the unit rod in order to identify unlabeled points; for example, if students are asked to identify 11 on the line above, they need to establish that, on this line, it is the red rod that is the unit length of 1. Other tasks require students to interpret what's wrong with "crazy number lines" that are partitioned into equal intervals, but the tickmarks are labeled incorrectly.

Throughout Lesson 5, students apply the principles of ***unit, multiunit, all numbers have a place but need not be shown, 0 has a place on the line,*** and ***order*** to support their reasoning.

Lesson structure. Lesson 5, like most LMR lessons, has a 5 part lesson structure. The sequence is designed to surface diverse student ideas, that the class resolves with appeals to number line principles and definitions. In this lesson, all principles learned to date are relevant as students identify missing numbers on non-routine number lines. These useful principles include: order, 0 is a number, unit interval, multiunit interval, and every number has a place but not all numbers need to be shown.



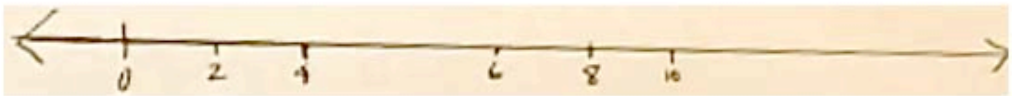
Patterns of student understanding. The 5 part lesson structure is designed to elicit and build on patterns of student understanding, by creating contexts of 'cognitive conflict' that require resolution. Findings from classroom studies have revealed that, in Lesson 5, students are challenged as they attempt to coordinate the ideas of *unit interval* and *multiunit interval,* as well as the ideas that 0 *is a number on the line* and *every number has a place but not all number needs to be shown.* Common patterns of partial understanding are shown below.



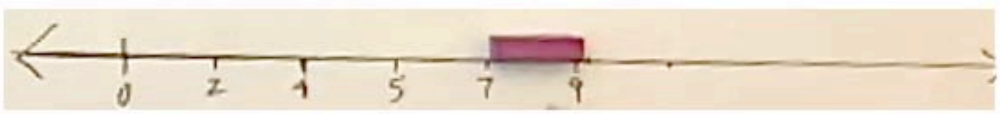
Guiding discussion and resolution. During Lesson 5, in whole class discussions and in partner work, the students and the teacher draw on the ideas about units, 0, and every number as a place as they reach agreements about mathematically reasonable interpretations and solutions to Lesson 5 tasks. A transcribed excerpt from one version of Lesson 5 is shown below.

Pedagogy: Pushing Student Thinking

T: What do you think of my number line?



Students: -You forgot the 5.
-It's fine. It's good!
-It's not, no....
T: Talk to the person next to you. What's wrong?
Students: -The space is too big between 4 and 6.
-It's not a unit interval.
T: Okay, I'll fix it. I'll be very careful when I use the rod to mark the line.



T: Okay, what's our multiunit interval?
Students: Two. But .....
T: Two okay. Between 7 and 9? 5 and 7? 2 and 4? 0 and 2?
Students: Yes, no, yes, no!! It's wrong! 4 and 5 is wrong!
T: Oh, *that's* my problem. Because once you have two numbers on the line, that interval has to stay the same *everywhere* on the line. If 7 to 9 is a multiunit of two, that distance is *always* two. So we can use that interval of 7 to 9 to find *any* other number on the number line!

Pre-Post Unit Findings

We analyzed evidence of student learning from gains on unit assessments in the classrooms of our collaborating teachers.

Sample

- 2 classrooms in 2 elementary schools in an urban school district
- Collaborating teachers with expertise in mathematics education
- 50 students in grades 4 (N=9) and 5 (N=41)

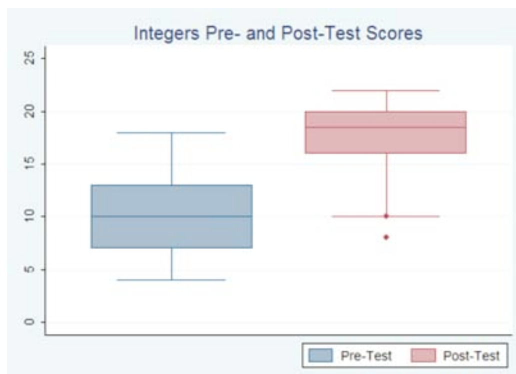
Unit Assessments

- Separate unit assessments for Integers and Fractions units.
- Integers Assessment: 22 items, 11 number line and 11 non number line; Reliability, alpha = 0.87.
- Fractions Assessment: 27 items, 16 number line and 11 non number line; Reliability, alpha = 0.92.

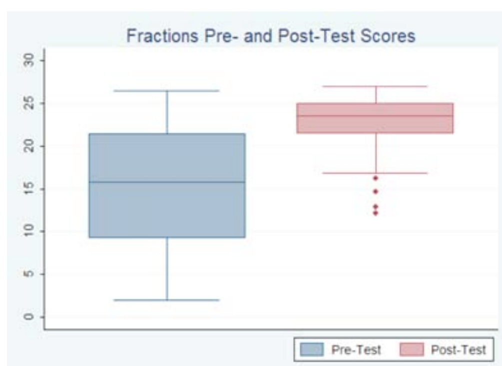
Results

Students made significant gains on both the Integers and Fractions unit assessments. Below we report findings for each unit, by grade, and for item type (number line vs. non-number line). The results for these classroom studies provide promising evidence of the utility and effectiveness of the LMR curriculum.

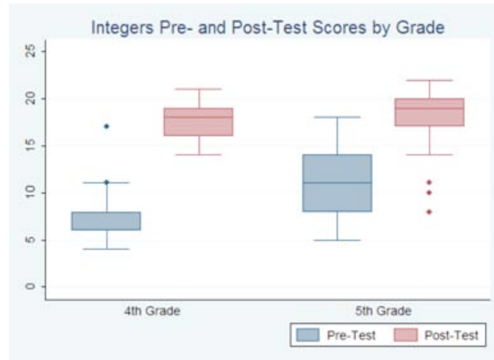
- Integers: Average gain was 7.3 points ($t(49) = 18.1, p < 0.001$).



- Fractions: Average gain was 7.5 points ($t(46) = 10.0, p < 0.001$).



By grade: Both 4th and 5th grade students made gains in each unit.



- Integers: Average gain for 4th graders was

  9.4; average gain for 5th graders was 6.9.

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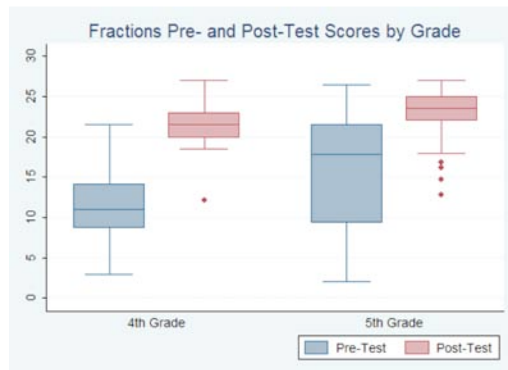
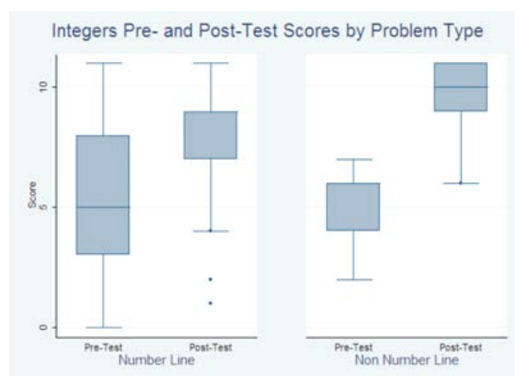
- Fractions: Average gain for 4th graders was

  10.0; average gain for 5th graders was 7.0.



By item type: Students made gains on both number line and non number line problems for both units.



- Integers: Average gain on number line

  problems was 2.7; average gain on non

  number line problems was 4.6.

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- Fractions: Average gain on number line

  problems was 5.6; average gain on non

  number line problems was 2.0.