Learning Fractions in a Linear Measurement Context: Development and Fieldtests of A Lesson Study Intervention

Catherine C. Lewis, Rebecca R. Perry, Shelley Friedkin and Elizabeth K. Baker, Mills College

The image of the U.S. curriculum as “a mile wide and an inch deep” is one of the best-known sound bytes to emerge from the Third International Mathematics and Science Study (TIMSS), and one that has prompted many calls for a more “coherent curriculum” in the U.S. (e.g., Schmidt, Houang, & Cogan, 2002, p. 3). Coherent curricula focus on a small number of key mathematical topics at any given grade level, treat each one in depth, and reintroduce the topic later only if it is developed in a more advanced manner (Schmidt, et al., 2002). For example, Japanese grade 8 mathematics textbooks include about 10 topics, while their U.S. counterparts include more than 30 topics (Schmidt, McKnight, & Raizen, 1997). Unfortunately, reducing the overload in US textbooks is not simply a matter of eliminating some topics. One reason that topics are repeated over the years of schooling is that some US students do not adequately master them when they are first taught, thus necessitating re-teaching. This problem is well-illustrated by fractions, a topic that is introduced *earlier* in the US than, for example, in Japan, but which continues to be taught and re-taught throughout the late elementary and middle school years in the US, long after basic fraction concepts have exited the curriculum in Japan.

Japanese students significantly outperform their US counterparts on the TIMSS mathematics assessment at both grade 4 and grade 8 in each administration (1995, 1999, 2003, 2007) (http://nces.ed.gov/timss). Six fourth grade fractions items from the 1995 TIMSS were released publicly, and Japanese students perform significantly above the international average on all six fraction items, with US students performing significantly above the international average on two items (TIMSS USA4, n.d.).

We became curious about the particular representations used to introduce fractions in US and Japanese textbooks, and whether these might shape students' subsequent understanding of fractions, a topic highlighted in some previous research (Watanabe, 2002, 2006, 2007) . The first part of this paper examines the representations used by two Japanese and two US textbook series to introduce and develop fractions. By fraction "representations" we mean any visual model or situation used to illustrate fractions. Types of representations include, for example, circle area (e.g., fraction of a pizza); part of a set (e.g., the fraction of red balloons in a bunch of red and blue balloons); and linear, area or volume measurement (e.g., fraction of a meter, rectangle, or liter). We ask how these representations might shape students' understanding of fractions. In some cases, the choice of representation may correspond to a particular meaning or interpretation of fraction that is the curricular focus. For example, five meanings of fractions are distinguished by the *Elementary School Teaching Guide for the Japanese Course of Study (Grades 1-6)(Takahashi, Watanabe, & Yoshida, 2004)*:

> There are several meanings of fractions. …
> To take up $\frac{2}{3}$ for example, this has the following meanings:
> 1. Representing two of three equally divided parts.
> 2. Representing the quantity resulting from a measurement such as $\frac{2}{3}$ meter or $\frac{2}{3}$ deciliter.
> 3. Representing twice the unit that is obtained when 1 is partitioned into three equal parts ($\frac{1}{3}$).
> 4. Representing the ration of A to B, i.e., the relative size of A when B is considered as 1.
> 5. Representing the quotient of 2 ÷ 3.
>
> These divisions are as a matter of convenience. Often some of these are used together when teaching.
>
> When fractions are introduced for the first time in this grade [3], the first two meanings, 1 and 2, are studied.
> Contrary to decimal numbers, which represent a specified unit such as $\frac{1}{10}$ or $\frac{1}{100}$, we can choose any fraction suitable for the unit such as $\frac{1}{3}$, $\frac{1}{4}$ or $\frac{1}{5}$. It is important to appreciate the significance of fractions such as this. However, though we can choose suitable forms of fractions as a unit, it is not easy to represent them on a number line. It is important to utilize equally spaced tick marks on a tape to help children acquire the size of $\frac{1}{3}$ or $\frac{1}{4}$, and to help children to gradually become able to express them on a number line *(Takahashi, et al., 2004).*

The different meanings of fractions are introduced at different points during elementary school. In the current mathematics *Course of Study*, fraction meanings 1-3 are introduced in grade 3, and meanings 4-5 are introduced in grade 5 (Takahashi, Watanabe, & Yoshida, 2008). Appendix A provides an overview of the fractions units in one of the Japanese textbook series we studied.

In the US, the NCTM's *Curriculum Focal Points* also notes, at grade 3, several meanings of fractions to be studied:

> Students develop an understanding of the meanings and uses of fractions to represent parts of a whole, parts of a set, or points or distances on a number line. They understand that the size of a fractional part is relative to the size of the whole, and they use fractions to represent numbers that are equal to, less than, or greater than 1. They solve problems that involve comparing and ordering fractions by using models, benchmark fractions, or common numerators or denominators. They understand and use models, including the number line, to identify equivalent fractions (NCTM, 2006, p. 15)
>
> Fraction as division quotient is treated in the grade 6 focal points (NCTM, 2006, p.18) and fraction as expression of probability is treated in grade 7 (NCTM, 2006, p.19).

The NCTM *Focal Points* and Japanese *Teaching Guide* are similar in the identification of several meanings of fractions, and in their choice of grade 3 for the major conceptual introduction of fractions. The two documents differ somewhat in the fraction meanings identified, as can be seen by comparison of the quotes above. The Japanese *Teaching Guide* does not specify part of a set as a fraction meaning, but does specify that non-unit fractions can be seen as accumulations of unit fractions (meaning 3), and that fractions express a quantity resulting from measurement. Although the NCTM curriculum does not distinguish measurement quantity as a meaning of fraction, it does discuss the role of linear measurement in building student understanding of fractions.

Students in grade 3 strengthen their understanding of fractions as they confront problems in linear measurement that call for more precision than the whole unit allowed them in their work in grade 2. They develop their facility in measuring with fractional parts of linear units.

We focus in particular on linear measurement representation of fractions, a representation that is not typically emphasized in U.S. textbooks, but is used successfully in some higher-achieving countries and in research-based curricula (Dougherty & Zilliox, 2003; Watanabe, 2007); ."Linear measurement" does not mean using marked rulers; rather, it means measuring the length of a long, narrow object by seeing how many times a smaller unit fits. As highlighted in Table 1 and as discussed below, linear measurement representations may help students see fractions as *amounts* (rather than primarily as *pieces* or as *descriptions of a situation*) and eventually to connect fractions to the number line. Japanese teachers' manuals emphasize the importance of building students' understanding of fraction as number, as the following excerpt illustrates:

> To develop understanding of the concept of fraction, it is important for teachers to attend to the following two aspects.
> 1. Fractions are useful to express an amount smaller than a standard unit of measure.
> 2. Like integers and decimal numbers, fractions have the characteristics of numbers.
> … If, as a result of third grade instruction, students develop a rigid way of thinking about fractions as so many parts of a partitioned whole, it can be difficult to then nurture an understanding of fraction as number (Tokyo Shoseki, 2000, 3B, p.85, translation by first author).

After the examination of textbooks, we describe a randomized controlled trial designed to support US teachers to investigate linear measurement representations of fractions and their potential affordances for students. The intervention is lesson study supported by a "resource guide" of materials including mathematical tasks, student work, curriculum, classroom video, research articles, and other materials designed to support teachers' investigation of fractions. In lesson study, teachers collaboratively study existing research and curriculum materials and plan, observe, and analyze a classroom "research" lesson that brings to life their ideas about how to teach a particular topic (in this case fractions). We compared small groups of teachers (each consisting of 4-10 teachers) randomly assigned to one of three conditions: lesson study using the fractions resource guide; lesson study without the resource guide; and professional development as usual (whatever occurs naturally at a site). Findings to be reported in this paper draw most heavily from the early phases of the research, in which we studied different curricular representations of fractions, developed the resource guide, tested it in small field studies with U.S. teachers (and their students), and gathered pre-intervention data from 39 groups of teachers that were subsequently randomly assigned to one of the study conditions.

(1) Examination of Elementary Textbooks from Japan and US

Table 2 shows the data resulting from review of fractions units in grades 1-5 textbooks from four textbook series, two from Japan and two from the U.S. The two Japanese textbook series include the most widely-used Japanese series (Hironaka & Sugiyama, 2006) as well as a second widely-used series (Hitotsumatsu, Okada, & Machida, undated). Both are available in English and the two series correspond to two different versions of the Japanese *Course of Study*, thereby providing windows into the Japanese curriculum during the 1990's (when the early TIMSS research was conducted) as well as during the first decade of the new century. The U.S. textbook series were also selected to provide two contrasting perspectives, that of a widely used mainstream textbook (Harcourt California Teachers' Edition, 2002) and a research-based, reform curriculum, *Investigations in Number, Data and Space* (Pearson Scott Foresman TERC, 2007). The central focus of this part of the investigation was to understand what representations of fractions are used in US and Japanese textbooks, in order to consider the differences and to identify ideas that might be more available to Japanese teachers and students than to their U.S. counterparts.

Table 2 lists the fraction representations used at each grade level in each textbook series, and it reveals several striking differences between the Japanese and

US textbooks. The US texts introduce fractions earlier (grade 1 or 2) than the Japanese texts (grade 3 or 4). Both Japanese texts use the same four representations (linear measurement, liquid measurement, number line, and rectangle area), whereas the two U.S. texts differ from each other, using a total of 15 representations between them, including clock, money, fraction bar, sets of objects and areas of various figures (circle, rectangle, and other polygons). We classified as a number line any representation that used ordered intervals on a line to represent numbers (see, for example, the line in Figure 3), even if the representations did not include the bi-directional arrows and marked zero point usually found on mathematicians' number lines.

One challenge in coding the textbooks was deciding what constitutes a linear measurement representation for fractions. For example, contrast the two textbook pages shown in Figures 2 and 3. Though they both use linear measurement (meters or inches) that entails fractions, the Japanese textbook example (Figure 2) uses linear measurement to establish the meaning of the fractions (i.e., to show the relationship between part and whole), whereas the US textbook example (Figure 3) uses linear measurement as part of reporting data in a bar graph (and the axis labels suggest prior understanding of fractions). To capture this difference, we subdivided the linear measurement category into two subcategories: use of linear measurement context to establish meaning of fractions vs. to report data.

Table 2 also shows that the US textbooks that we studied do not use linear measurement to establish the meaning of fractions, and that the Japanese texts do not use at all several fractions representations used extensively in one or both US texts, including area of circle and part of a set. Three representations are used in both countries (in at least one textbook): volume measurement, number line, and rectangle area. The Japanese textbook explicitly links the different fraction representations that are used. For example, Figure 1 shows how one Japanese textbook links area and number line representations; this type of linking did not occur in the US texts. The teachers' manual for the Japanese textbooks highlights the importance of having students move from concrete to abstract understanding of fractions, likening this process to how students learned about integers:

> Students learned to think about whole numbers by replacing concrete materials such as apples or papers with abstract materials (pictures of squares and tape diagrams), then comparing them on the number line. The same approach can also be used to help students understand fractions as numbers. This textbook unit first introduces fractions using an amount of water [in a square beaker], then replaces this with a picture of a square, and finally moves to expressing the amount on a number line. (Tokyo Shoseki 4b, p. 46)

In summary, the US textbooks introduce fractions earlier, and use a total of 15 representations across the two US texts, compared to four representations in the two Japanese texts. One central representation in both Japanese texts (linear measurement as a way to establish fraction meaning) is not used at all in the US texts, and two central representations in the US texts (part of a set and area of a circle) are not used at all in the Japanese texts. Why is linear measurement emphasized in the Japanese texts, and how might this representation shape students' images and understandings of fractions? In Table 1, we lay out some tentative hypotheses about the affordances of the linear measurement representation for developing an understanding of fractions. While it is likely that each representation of fractions has particular affordances, we chose to focus on the linear measurement representation because it is *not* a core representation in the US texts under study, and therefore represents a potential source of new affordances for US learners.

(2) Investigating a Linear Measurement Context for Fractions: A Resource Guide for Lesson Study

Lesson study is a professional development approach that originated in Japan, but has spread widely in North America and other parts of the world (Lewis, Perry, & Hurd, 2009) As shown in Figure 4, lesson study consists of cycles of collaborative activity in which a team of teachers studies the curriculum; plans a "research lesson" that is taught by one team member while the others observe students and collect data; and discusses the data collected during the lesson, drawing out the implications for teaching and learning of the topic and more broadly (Lewis, 2002; Wang-Iverson & Yoshida, 2005).

We developed a resource guide designed to support elementary-school lesson study groups interested in the study of fractions. Table 3 shows the table of contents for the Fractions Resource Guide, which was developed through study of existing research and curriculum materials and progressively refined through two field studies, described in the next section. The Resource Guide was designed to support teachers' work during especially the first two phases of lesson study, in which teachers study available curriculum, research, and other resources and plan a research lesson. As Table 3 shows, the Resource Guide provided fraction problems for teachers to solve and discuss, student solutions to analyze, and resources (including a Japanese textbook series and lesson videos and plans) teachers could use to investigate linear measurement representations of fractions.

Based on the thinking laid out in Table 1, the resource guide focused in large part on linear measurement representations of fractions, with the expectation that the linear measurement context would provide a useful and, in many cases, new, perspective on fraction meaning. In addition, the Resource Guide provided basic materials related to lesson study, such as sample meeting agendas, a template for developing a lesson plan, and protocols for observation and discussion of the research lesson. For the randomized controlled trial, two versions of the Resource Guide were developed: one version included the materials on fractions as well as materials to support lesson study; the other version included just the tools designed to support lesson study.

(3) Field Studies and Development of Assessments

In order to develop and test Resource Guide materials and assessment items, two small field studies were conducted. In the first field study, an experienced Japanese elementary teacher served as guest instructor of a series of three public research lessons to grade 3-5 students at a small independent school in California, using a linear measurement context to introduce fractions, based on a Japanese textbook unit (Hironaka & Sugiyama, 2006). A group of six US educators collaborated closely with the Japanese instructor to collect and discuss data from each lesson, and 20-30 additional educators observed the public research lessons, taking part in post-lesson discussions each day, based on lesson study protocols. Video of the research lessons and post-lesson discussions, lesson artifacts, student written work, and prepost- clinical interviews of four students were collected. The lesson plans and a summary of the three fractions lessons are provided in Appendix B.

Post-lesson discussions and written reflections from the three research lessons provided data on the reactions of local teachers to linear measurement representations of fractions. Teachers' comments strengthened our interest in focusing on a linear measurement context for fractions, since many teachers found this to be a new idea, but one with close connections to their own problems of practice, as the following quotes suggest:

> The objective of learning fractions with measurement was an important "aha" for me. In American math curric [sic] – we tend to teach child about parts of wholes – not in relation to a specific measurement that stays constant. Using a unit like meter that stayed constant helped the students solidify their conceptual understanding of parts of a whole. [day 1 reflection]
>
> The question of linear versus a "pie" understanding was really compelling for me. It's a distinction in the concept of fractions that I hadn't considered and I wonder what my own understanding of fractions would be like if I had been first introduced that way. [day 2 reflection]
>
> I see the benefits of understanding fractions as part of a unit. Based on the unit, $\frac{1}{2}$ is a different measurement. With the pie version one half becomes a fixed unit not a variable unless there is $\frac{1}{2}$ of different size pies. I also like its potential to tie into measurement/number line. [day 2 reflection]
>
> The way American schools have traditionally taught fractions is by using circles, pies, pizzas, etc. I have never heard of introducing fractions through linear measurement. The idea of starting with a unit (e.g., meter) and having students explore fractions in this manner is very interesting and new for me. This lesson helped broaden my own understanding of fractions by seeing them as parts of a whole and numbers. [day 2 reflection]
>
> I like how the linear model contributes to understanding a unit. I like the molding of measurement to number line. I saw a student who thought the "small bit" was $\frac{1}{8}$ then go to $\frac{1}{10}$, making a conceptual leap from size of denominator to size of fraction. [day 3 reflection]

Another source of data from field study 1 were clinical interviews from four students (two each from grades 3 and 5). (The classroom teacher chose four students expected to be comfortable being filmed and interviewed by unfamiliar researchers.) The grade 5 students had previously studied fractions, whereas the grade 3 students had not previously had a formal introduction to fractions. Students were given a paper and pen and asked to explain what "$\frac{3}{5}$", "$\frac{7}{5}$" and "$1\frac{4}{5}$" were, in order. A comparison of the interviews before and after the series of lessons reveals no changes in the representations the children spontaneously used to show the fractions. Both fifth grade students were able to locate the given fractions on a number line and used circle area representations both before and after the series of lessons. One fifth-grade student also used a rectangle area representation both before and after the lessons, and used a set representation before the lessons. One of the third-grade students drew a circle and divided it, and the other (who could not correctly represent the fraction) drew 5 groups of 3 objects each. (We could not tell whether the student was attempting to show a set representation for $\frac{3}{5}$ or was recalling information about multiplication or division.)

When asked to show unit fractions in relation to a drawn line labeled as 1 meter, the two third grade students used the linear measurement representation to demonstrate an understanding of the relationship between a unit fraction and the whole (i.e., that *x* number of $\frac{1}{x}$ units comprises the whole; see student work in Appendix C). Both students "measured" by making a space between finger and thumb that fit 3 times into the "meter", in order to show that $\frac{1}{3}$ goes into a meter 3 times;

when asked to show $\frac{1}{4}$ meter, they decreased the space between finger and thumb to show that $\frac{1}{4}$ goes into a meter 4 times. These actions suggest that the third grade students grasped the relationship between fractional part and whole, and their actions also provide useful information about the gestural cues that may reveal student thinking. Information about student thinking can play a key role in improvement of teaching (Franke, Carpenter, Levi, & Fennema, 2001). Appendix D provides photographs and transcript material from one third-grade student, showing her gestures.

In summary, the findings from the clinical interviews indicated that four children within the same school initially used several different representations when asked to show a fraction, and none of the children spontaneously chose to use the linear measurement representation, even after taking part in the series of three lessons introducing fractions in a linear measurement context. However, the younger students demonstrated some new understanding of the part-whole relationship when using a linear measurement representation presented to them, since they created an interval with their fingers and used it to measure the whole. These results sparked our interest in the potential importance of the first representations students learn–since none of the students shifted representations–and in the experiences that might enable students to connect different representations. The latter question arose because the third grade students (neither of whom could represent $\frac{3}{5}$ without assistance in the pre-interview, and both of whom described the fraction as something related to "division" or something you could "sort", (perhaps referring to fair-share division) showed an understanding of the relationship between a unit fraction and the whole when *given* a linear context in the interview following the series of lessons. (But they did not spontaneously use the linear measurement context to make sense of fractions in other interview questions.) As a result of the clinical interviews, we realized that supporting students to understand a linear measurement representation and connect it with other fraction representations was a substantial undertaking, and we expanded sections of the Resource Guide in which teachers notice, share and discuss the fraction representations used in their own problem-solving as well as consider the representations students might use.

Field Study 1 also yielded "actionable artifacts" (Bannan-Ritland, 2003) the public research lessons, including lesson plans, video of the instruction, and examples of student thinking; these materials became central to the Resource Guide, and to the work of the teachers in Field Study 2.

In Field Study 2, a lesson study group consisting of a mathematics coach and four elementary teachers from an urban, high-poverty school used a draft version of the Resource Guide (including the analytic framework, the Japanese curriculum, and video, lesson plans, and other materials from the public research lessons) to design a research lesson for grade 2 students, in which students were introduced to fractions using a linear measurement representation. Video from this lesson study cycle shows students using linear measurement representation tasks to develop some of the understandings in Table 1 (e.g., the idea that a whole meter is four times $\frac{1}{4}$ meter, and that a denominator with a larger numerical value has a smaller size.) These students had previously studied fractions using area and set representations. Their struggle, focused conversations, and excitement surrounding what to call a piece that goes into a meter a certain number of times all suggested they were encountering a new perspective on fractions, and not simply practicing the part-whole representation that was already familiar to them. Video of the research lesson from Field Study 2 was incorporated into a new version of the Resource Guide to be used in the next study.

Taken together, Field Studies 1 and 2 also helped us identify aspects of teacher and student thinking about fractions to be assessed in the subsequent intervention study. For example, a conversation following the third research lesson of Field Study 1 highlighted the potential affordances of the linear measurement representation for seeing the multiplicative relationship between the unit fraction and the whole (that $\frac{1}{x}$

goes x times into 1, or, as in the conversation below, that $\frac{2}{5}$ goes $2\frac{1}{2}$ times into 1).

The following conversation occurred during the post-lesson discussion of the third fraction lesson taught by the Japanese instructor. Appendix E illustrates some of the student thinking that prompted the teacher's comment.

> Teacher on Lesson Study Team: I am wondering how student thinking evolves about…the relationship between how many times something goes in and what we call that piece: two and a half times, versus two-fifths
>
> Visiting Japanese Instructor: That is exactly the relationship between multiplication and division. So how will [having] the concept of multiplication and division affect [students' understanding of] fractions? To me, teaching fractions before multiplication and division doesn't make sense. Teaching multiplication and division should be a foundation of teaching fractions. Because 'equally dividing into' is the foundation of making a unit fraction.

The next section provides examples of items we located or developed and tested during the period of the Field Studies in order to investigate students' and teachers' understanding of the multiplicative relationship between the unit fraction and the whole (for example, $\frac{1}{4}$ times 4 is 1) and between unit fractions and non-unit fractions with the same denominator (for example, $\frac{1}{4}$ times 3 is $\frac{3}{4}$). These items ask respondents to relate unit fractions to non-unit fractions and wholes, areas suggested by the Field Studies as potentially important areas of impact of the linear measurement representation of fractions.

(4) Randomized, Controlled Trial

Building on the findings of Field Studies 1 and 2, we designed a randomized controlled trial to evaluate the effectiveness of lesson study supported by the Fractions Resource Guide. Here, we present results only from the pre-intervention phase of that study. Pre-tests administered to 214 teachers who would later be randomly assigned to one of the study conditions provide baseline data regarding teachers' facility with the linear measurement representation of fractions. The participating teachers came from 27 districts across the United States, representing a wide range of demographic circumstances. Several items asked teachers to solve fraction problems posed in a linear measurement context. One ("Highway Length") was the following:

> A highway is under construction. The workers have completed $\frac{2}{5}$ of the total length. If the workers complete an additional 7.5 miles, then they would have completed $\frac{1}{2}$ of the total length. What is the total length (miles) of this highway under construction? (Zhou, Peverly, & Xin, 2006)

Another ("Make $\frac{3}{4}$ unit") was as follows:

> The line segment below has length $\frac{9}{8}$ unit. Show by drawing and explain how to create a line segment of length $\frac{3}{4}$ unit. (Adapted from Beckmann, 2005)

$(\frac{9}{8})$

Table 4 provides data on selected items from the teacher pre-assessment. As it shows, a substantial number of teachers did not provide correct answers to these items, although item features other than the linear measurement context may have contributed to teachers' difficulty with the items. Appendix F provides examples of the most common correct and incorrect solution strategies for creating a $\frac{3}{4}$ unit from a $\frac{9}{8}$ unit. Most teachers who solved this problem correctly used one of the two strategies shown in Appendix F, dividing the $\frac{9}{8}$ into pieces of $\frac{1}{8}$ unit or into pieces of $\frac{3}{8}$ unit. Most teachers who solved it incorrectly misidentified the whole–they found $\frac{3}{4}$ of the $\frac{9}{8}$, rather than $\frac{3}{4}$ of the base unit, as also shown in Appendix F.

Several open-ended items assessed whether teachers spontaneously used linear representations in response to hypothetical teaching situations. Relatively few teachers (13%) spontaneously proposed use of a linear representation in response to the following teaching situation (developed by our group):

> Your colleague's students are struggling with the problem: $\frac{3}{4}+\frac{1}{2}+\frac{5}{6}=2\frac{1}{12}$. Your colleague asks you to teach a lesson to her students to help them understand why this number sentence is true. Briefly outline the lesson you might design, describing what you would do at each step of the lesson. Specify what you would hope to find out about students and how you would find it out.

Likewise, although 94% of teachers correctly responded to the following item by identifying $\frac{7}{3}$ as a possible fraction, only 6% of teachers proposed to use a linear measurement representation to work with Anna:

> Anna says $\frac{7}{3}$ is not possible as a fraction.
>
> a) Is $\frac{7}{3}$ possible as a fraction? Yes No (Circle one.)
>
> b) What action, if any, do you take as a teacher to respond to Anna? (Ward & Thomas, 2009)

These data suggest that teachers may not see the usefulness of the linear measurement representation, nor use it spontaneously.

Pre-tests were also administered to one class of students from each of the groups participating in the study, totaling more than 2,000 grade 2-6 students across the 39 sites. Three assessment forms were developed for students in grades 2-3, 4, and 5, with easier items shared across grade levels and more difficult items added for the higher elementary grades. Appendix G includes some of the items that use a linear measurement context. Results shown in Table 5 include only currently available student pretest data from 5 study sites in four states (N=141).

Table 5 suggests a substantial gap between students' understanding of the fraction item (bear) and the corresponding integer (rabbit) item. For the grade 6 students, the number line context seems to have been more challenging than problems posed only with numbers and words.

(5) Conclusions

Although data analyses are still in progress, our work to date indicates differences in the representations used to introduce fractions in the two US and two Japanese textbook series we studied, as prior research suggested (Watanabe, 1996, 2001, 2002, 2006, 2007). There are a total of four representations in the Japanese books and 15 in the US books. Whereas the philosophy underlying the two US texts might be summarized as "students will learn from seeing as many different representations of fractions as possible," the philosophy of the two Japanese texts might be expressed as "establish fraction meaning using measurement, and then explicitly connect measurement representations to the number line." We wonder how each approach might impact students' development of stable, generalizable ideas about fractions, and also how the number of representations might support (or impede) teachers' development of mathematical knowledge for teaching (Ball, Hill, & Bass, 2005). We can imagine arguing either side of the case: that multiple representations help or confuse students in the early stages of understanding fractions. A small number of representations may enable students and teachers to develop robust, shared objects for discussion and reflection. On the other hand, a variety of representations may highlight the shared number that underlies many different concrete examples (e.g., the similarity between $\frac{2}{3}$ cup and 2 out of 3 pieces of pizza).

As noted, the textbooks also differ with respect to the particular representations chosen. Linear measurement is used as a context to establish the meaning of fractions in the Japanese texts but not the US texts. Table 1 lays out our conjectures about the affordances of linear measurement representations for helping students use length to establish a strong image of fraction size, to grasp fractions as numbers, and to make links to the number line. As noted above, grade 4 Japanese students scored significantly above the international average on all six fractions items released for 1995 (US students scored significantly above the international average for two items). Interestingly, the Japanese students significantly outperformed US students on the item shown in Figure 6 (the only released item using a circle area representation), despite the fact that this representation seems to be found in US textbooks but not Japanese textbooks.

Our studies to date also suggest that US students can use the linear measurement context to build an understanding of certain key ideas (i.e., that $\frac{1}{x}$ goes x times into the whole, and that $\frac{1}{x}$ is larger than $\frac{1}{x+1}$). The field studies also suggest that students may find it difficult to represent fractions in a linear measurement context, and the linear measurement context is unfamiliar to many US teachers as a way to introduce or explain fractions, although teachers see potentially useful connections to problems of practice. Unless we imagine that the rather dramatic differences between US and Japanese textbooks in their representations of fractions are inconsequential, it behooves us to investigate them further, and to understand how they might support or impede a coherent curriculum.

References

Ball, D. L., Hill, H., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator, 29*(3), 14-17, 20-22, 43-46.

Bannan-Ritland, B. (2003). The role of design in research: The integrative learning design framework. *Educational Researcher, 32*(1), 21-24.

Beckmann, S. (2005). *Mathematics for elementary teachers*. Boston: Pearson Education, Inc.

Dougherty, B. J., & Zilliox, J. (2003). Voyaging from theory to practice in teaching and learning: A view from Hawai'i. In N. Pateman, B. J. Dougherty & J. Zilliox (Eds.), *Proceedings of the 2003 joint meeting of PME and PMENA* (pp. 17-31). Honlulu, HI: Curriculum Research & Development Group, University of Hawai'i

Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal, 38*(3), 653-689.

Hironaka, H., & Sugiyama, Y. (2006). *Mathematics for Elementary School, Grades 1-6*. Tokyo: Tokyo Shoseki Co., Ltd. (English language version available at www.globaledresources.com).

Hitotsumatsu, S., Okada, Y., & Machida, S. (undated). *Mathematics for Elementary School*. Tokyo: Gakkotosho Co, Ltd.

Lewis, C. (2002). *Lesson Study: A handbook of teacher-led instructional change*. Philadelphia, PA: Research for Better Schools.

Lewis, C., Perry, R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*(4), 285-304.

National Council of Teachers of Mathematics (NCTM) (2006). *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence*. Reston, VA.

Pearson Scott Foresman TERC (2007). *Investigations in Number, Data and Space*. Boston: Pearson Education.

Schmidt, W. H., Houang, R., & Cogan, L. (2002). A coherent currriculum: The case of mathematics. *American Educator*, 1-18.

Schmidt, W. H., McKnight, C., & Raizen, S. (1997). *A splintered vision: An investigation of US mathematics and science education*. Dordrecht: Kluwer.

Takahashi, A., Watanabe, T., & Yoshida, M. (2004). Elementary school teaching guide for the Japanese Course of Study: Arithmetic (Grades 1-6): Madison, NJ: Global Education.

Takahashi, A., Watanabe, T., & Yoshida, M. (2008). English Translation of the Japanese Mathematics Curriculum in the Course of Study (Grades 1-9). Retrieved March 1, 2010, from http://www.seiservices.com/APEC/APEC_KB/KBDisplay.aspx?lngPkID=1567

TIMSS USA4 (n.d.). Mathematics Items Retrieved March 30, 2010, from http://nces.ed.gov/timss/pdf/TIMSS4_Math_Items_1.pdf

Tokyo Shoseki Co. Ltd. (2000). *Atarashii sansuu: Kyoushiyou shidousho kenkyuuhen [New mathematics: Teachers guide, research volume]* Tokyo: Tokyo Shoseki Co., Ltd.
Wang-Iverson, P., & Yoshida, M. (2005). *Building our understanding of lesson study*. Philadelphia: Research for Better Schools.
Ward, J., & Thomas, G. (2009). From presentation on teaching scenarios. Retrieved June 1, 2009, from http://www.nzmaths.co.nz/fractions
Watanabe, T. (1996). Ben's understanding of one-half. *Teaching Children Mathematics, 2*(8), 460-464.
Watanabe, T. (2001). Let's eliminate fractions from primary curricula! *Teaching Children Mathematics, 8*(2), 70-72.
Watanabe, T. (2002). Representations in teaching and learning fractions. *Teaching Children Mathematics, 8*(8), 457-463.
Watanabe, T. (2006). The teaching and learning of fractions: A Japanese perspective. *Teaching Children Mathematics, 12*(7), 368-374.
Watanabe, T. (2007). Initial treatment of fractions in Japanese textbooks. *Focus on Learning Problems in Mathematics, 29*(2), 41-60.
Zhou, Z., Peverly, S. T., & Xin, T. (2006). Knowing and teaching fractions: A cross-cultural study of American and Chinese mathematics teachers. *Contemporary Educational Psychology, 31*(4), 438-457.

Figure 1: Connection of Fraction Models (Hironaka & Sugiyama, 2004)

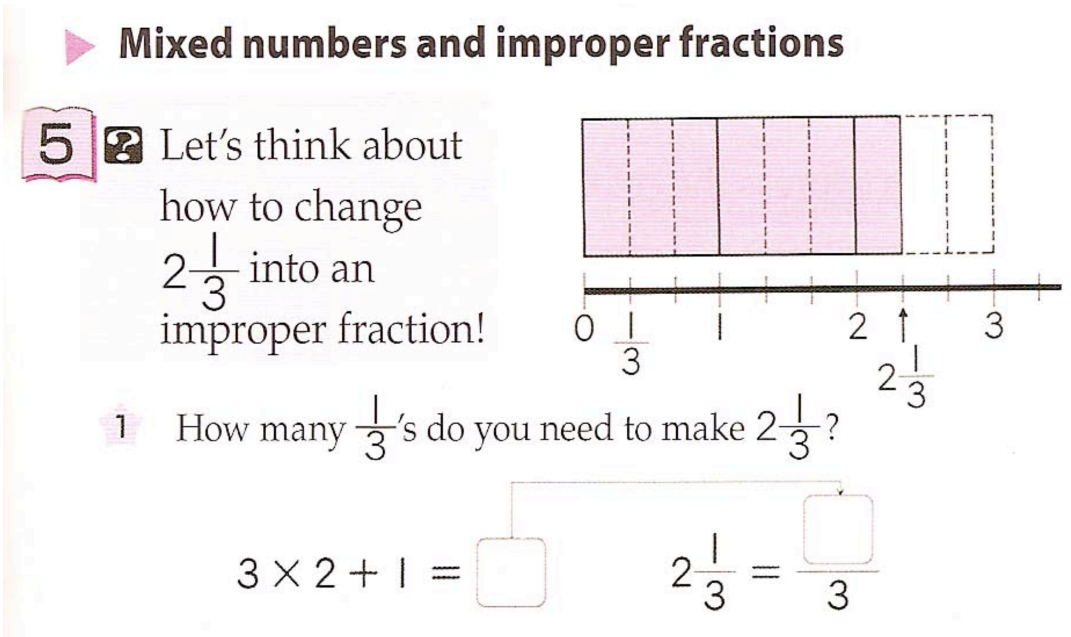
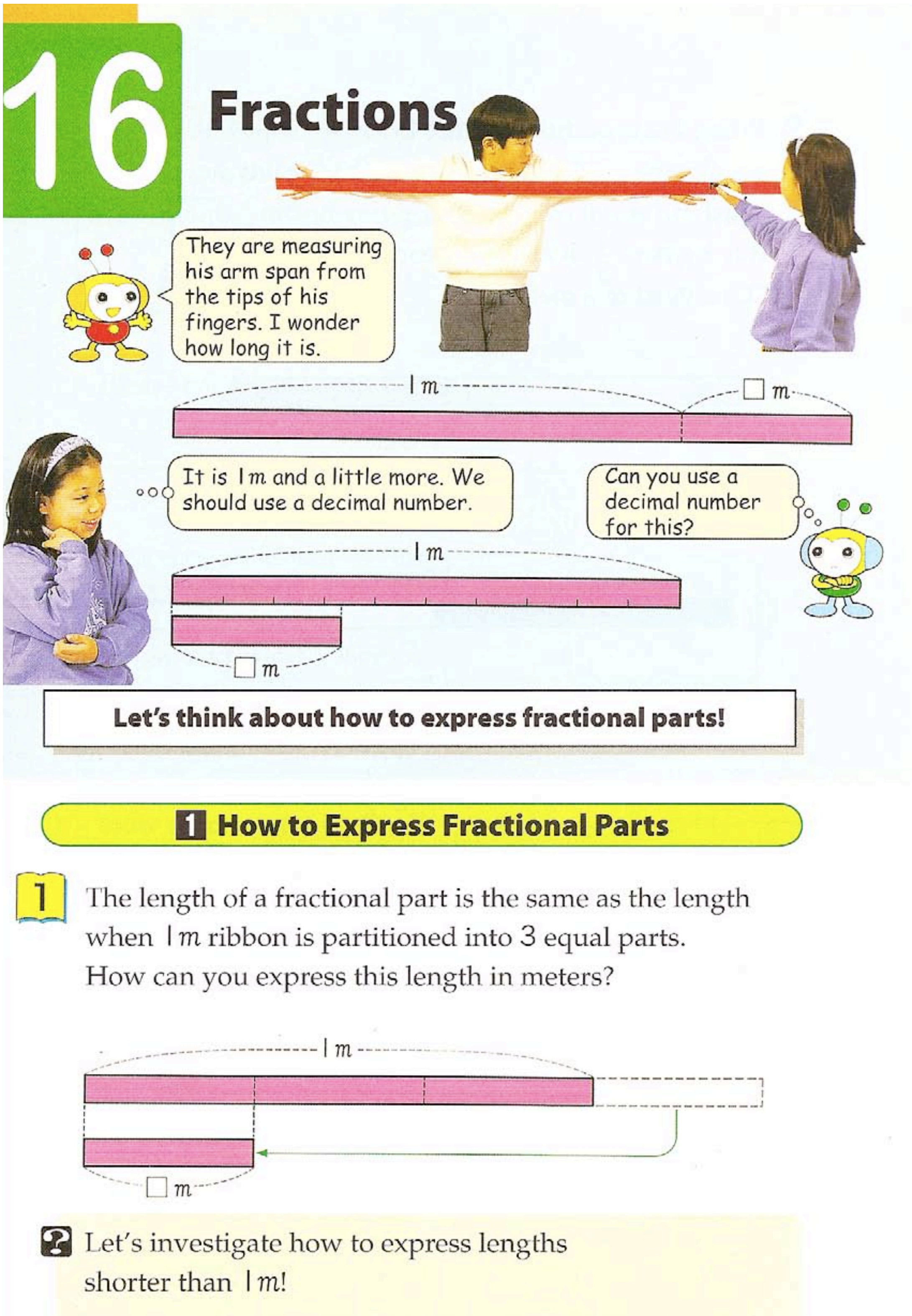


Figure 2: Linear Measurement to Introduce and Establish Meaning of Fractions
(Hironaka & Sugiyama, 2004, grade 3)



16 Fractions

They are measuring his arm span from the tips of his fingers. I wonder how long it is.

1 m
□ m

It is 1 m and a little more. We should use a decimal number.

Can you use a decimal number for this?

1 m
□ m

Let's think about how to express fractional parts!

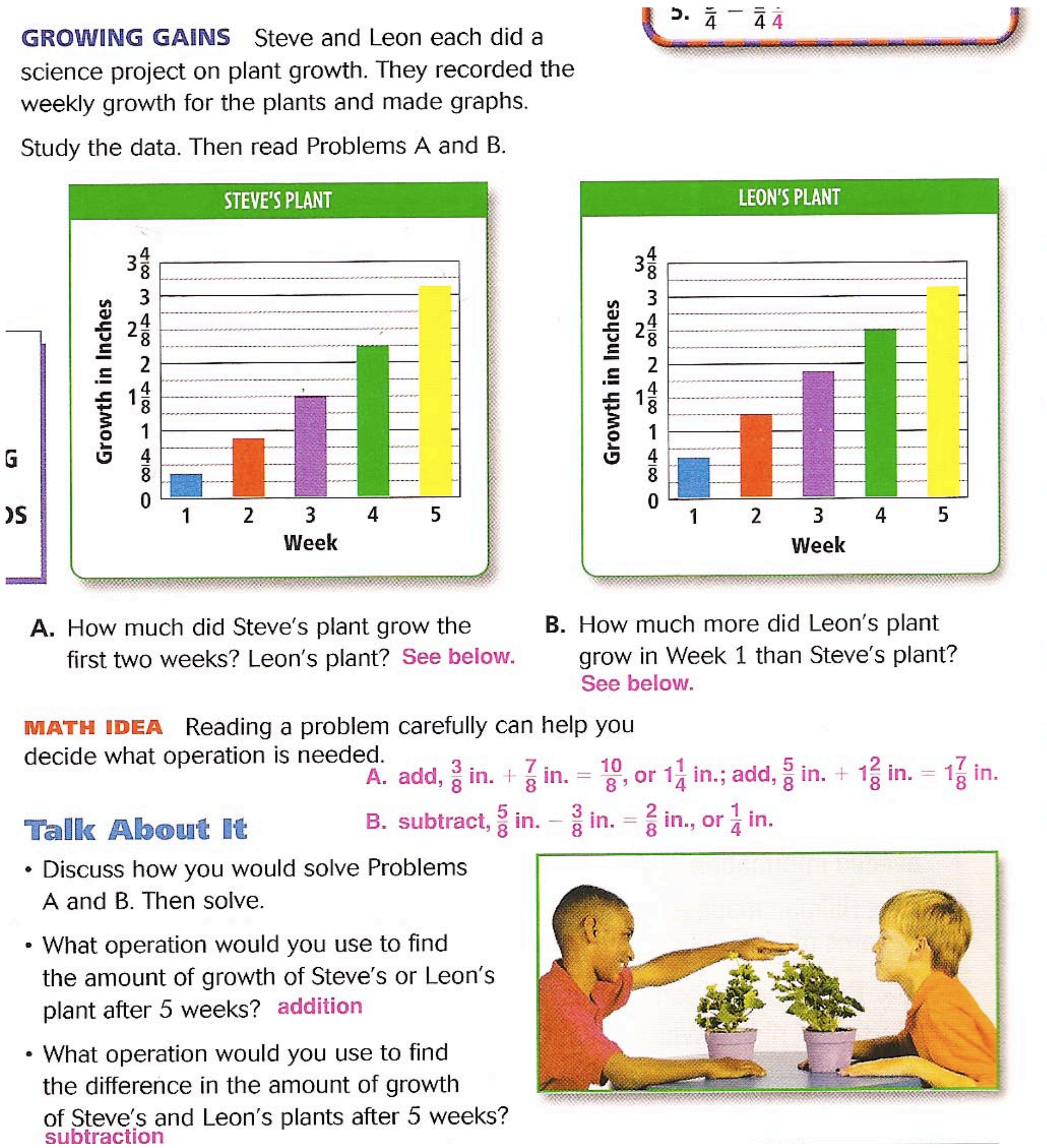
1 How to Express Fractional Parts

1 The length of a fractional part is the same as the length when 1 m ribbon is partitioned into 3 equal parts. How can you express this length in meters?

1 m
□ m

Let's investigate how to express lengths shorter than 1 m!

Figure 3: Linear Measurement in Data Report (Harcourt California)



GROWING GAINS Steve and Leon each did a science project on plant growth. They recorded the weekly growth for the plants and made graphs.

Study the data. Then read Problems A and B.

A. How much did Steve's plant grow the first two weeks? Leon's plant? See below.

B. How much more did Leon's plant grow in Week 1 than Steve's plant? See below.

MATH IDEA Reading a problem carefully can help you decide what operation is needed.

A. add, $\frac{3}{8}$ in. + $\frac{7}{8}$ in. = $\frac{10}{8}$, or $1\frac{1}{4}$ in.; add, $\frac{5}{8}$ in. + $1\frac{2}{8}$ in. = $1\frac{7}{8}$ in.

B. subtract, $\frac{5}{8}$ in. − $\frac{3}{8}$ in. = $\frac{2}{8}$ in., or $\frac{1}{4}$ in.

Talk About It

- Discuss how you would solve Problems A and B. Then solve.
- What operation would you use to find the amount of growth of Steve's or Leon's plant after 5 weeks? addition
- What operation would you use to find the difference in the amount of growth of Steve's and Leon's plants after 5 weeks? subtraction

Figure 4: Lesson Study

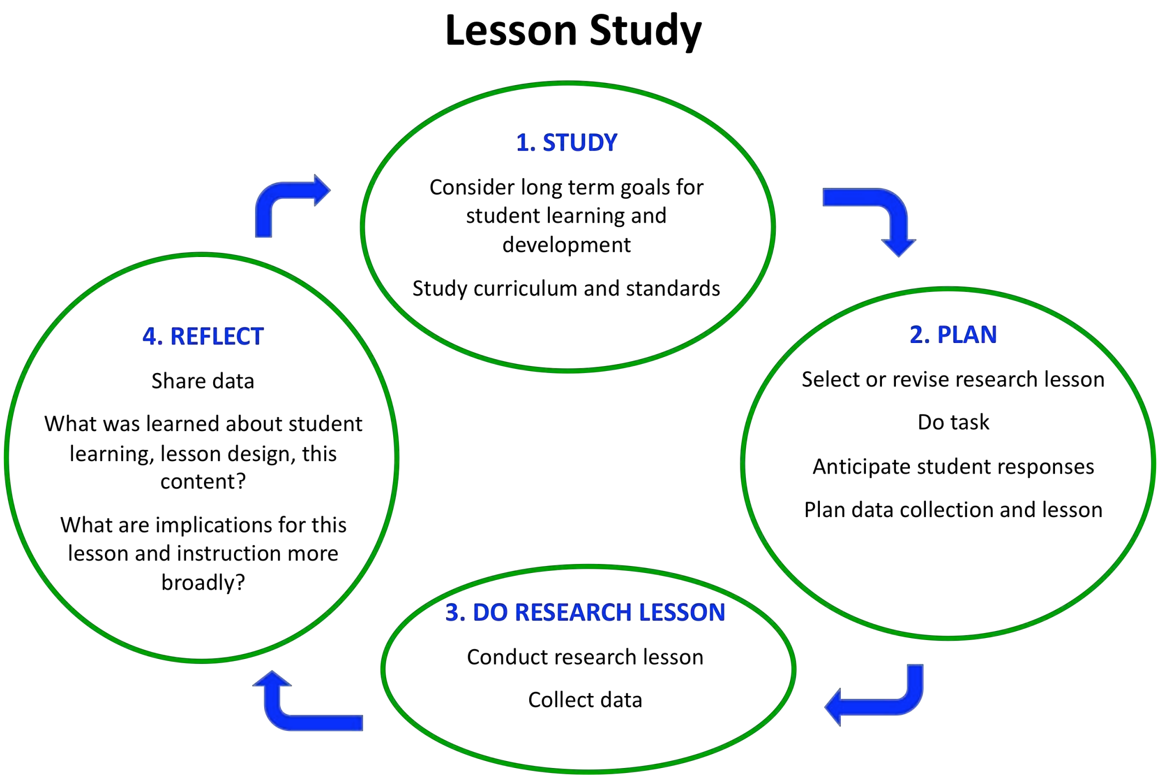


Figure 5: TIMSS Item

TIMSS 1995 4th-Grade Mathematics Concepts and Mathematics Items

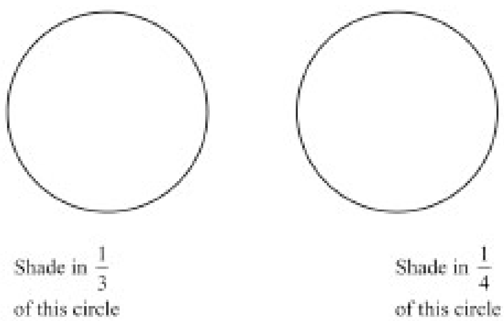
Content Domain	Cognitive Domain
Fractions and Proportionality	Solving Problems

Fractions of pie

Sam said that $\frac{1}{3}$ of a pie is less than $\frac{1}{4}$ of the same pie.

Is Sam correct? ________________

Use the circles below to show why this is so.



Overall Percent Correct

Singapore	58	▲
Netherlands	56	▲
Hong Kong	47	▲
Japan	41	▲
Korea	41	▲
Israel	39	▲
Hungary	33	O
Czech Republic	31	O
Ireland	30	O
Austria	27	O
United States	27	O
International average	**26**	
Slovenia	26	O
Australia	24	O
Latvia (LSS)	24	O
Iceland	22	O
Canada	20	O
Cyprus	20	▼
England	20	▼
Scotland	20	▼
New Zealand	16	▼
Norway	16	▼
Greece	14	▼
Thailand	14	▼
Kuwait	6	▼
Portugal	6	▼
Iran, Islamic Rep.	3	▼

Country average vs. International average:

Higher ▲
Not different O
Lower ▼

Item Number: V1

SCORING

Note: The partition of circles has priority over shading. This is reflected in the scoring guide below.

Correct Response

- NO. Both circles are correctly partitioned.

Partially Correct

- NO. No partitioning is shown.
- NO. Only one of the circles correctly partitioned.
- NO. Other incorrect ways of partitioning.
- YES, or there is no conclusion stated. Both circles are correctly partitioned.
- Other partial.

Incorrect Response

- YES. No partitioning is shown.
- YES. The part representing 1/3 is made consistently smaller than the part representing 1/4.
- YES. Other responses where one or both of the circles partitioned into 3 and/or 4 parts.
- Other incorrect.

Table 1: Conjectures About Affordances of Linear Measurement Context For Understanding Fractions

Student Understanding	Support From Linear Measurement Context?
Fraction As Number A fraction represents an amount, *not* just a counted number of pieces (such as 2 of 3 pieces of a pizza) or a situation (such as 2 of 3 shirts are red).	Students may think about relative length ("how much" or "how long") rather than just counting the number of pieces ("how many").
The Meaning and Size of the Denominator • Different units (such as $\frac{1}{3}$ and $\frac{1}{5}$) are different sizes. • The more units a whole is partitioned into, the smaller each one is. • $\frac{1}{n}$ fits exactly n times into the whole.	Compared to area, which can be divided in many ways, length may provide a stronger image of the relative size of $\frac{1}{3}m$, $\frac{1}{2}m$, etc., as well as a stronger image that the unit that fits in 3 times is $\frac{1}{3}$ that $\frac{1}{6}$ is half the length of $\frac{1}{3}$, etc.
Knowing What is the Whole • Constructing the whole when given a fractional part. • Keeping track of the whole.	It may be easier to visualize and keep track of the whole when using a familiar standard measurement unit (e.g., meter), rather than an ad hoc unit (e.g., pie pieces).
Fraction Size Seeing a non-unit fraction as an accumulation of unit fractions. [A unit fraction has a numerator of 1; a non-unit fraction has a numerator other than 1.]	When students think about a turtle who travels in a straight line $\frac{1}{5}$ mile a day for 4 days, they may easily develop an image of $\frac{4}{5}$ as four fifths —as $\frac{1}{5}$, repeated four times. In contrast, $\frac{4}{5}$ of a rectangle or circle may not provide the same strong image of repetition of $\frac{1}{5}$ since the area can be split in many different ways.
Fractions Can Represent Quantities Greater Than One May be difficult for students who have a strong image of a fraction as a *piece* of something.	When students measure an object that is longer than 1 foot (meter, etc.), it may be relatively easy to visualize something as a whole plus an additional fractional part and understand the meaning of fractions greater than 1.

Table 2: Fraction Representations in Four US and Japanese Textbooks

	US: Harcourt	US: Investigations	Japan: TS	Japan: GT
Grade 5	Volume Measure Number Line Area: Rectangle Set Fraction bar, strip Bar Graph Money Linear Measure: Data Report 3-D Objects Weight	Number Line Area: Rectangle Area: Circle Set Clock Linear Measure: Data Report Fraction track	Linear Measure: Fraction Meaning Volume Measure Number Line Area: Rectangle	Linear Measure: Fraction Meaning Volume Measure Number Line Area: Rectangle
Grade 4	Volume Measure Number Line Area: Rectangle Area: Circle Area: Other Figure Set Fraction bar, strip Weight Bar Graph Linear Measure: Data Report Music Notes Money	Volume Measure Number Line Area: Rectangle Set Money	Linear Measure: Fraction Meaning Volume Measure Number Line Area: Rectangle	Linear Measure: Fraction Meaning Volume Measure Number Line Area: Rectangle
Grade 3	Volume Measure Number Line Area: Rectangle Area: Circle Area: Other Figure Set Fraction bar, strip Weight Money Linear Measure: Data Report	Area: Rectangle Area: Circle Area: Other Figure Set Weight Money	Linear Measure: Fraction Meaning Volume Measure Number Line	
Grade 2	Volume Measure Number Line Area: Rectangle Area: Circle Area: Other Figure Set Fraction bar, strip	Volume Measure Area: Rectangle Area: Circle Area: Other Figure Set 3-D Object Volume Linear Measure: Data Report		
Grade 1	Area: Circle Area: Other Figure Set			

Table 3: Fractions Resource Guide Table of Contents

Section 1: Mathematics Tasks to Solve and Discuss
Student Responses for Mathematical Tasks
Daily Meeting Reflection Forms
Section 2: Curriculum Inquiry: Different Models of Fractions
Developing a Number Sense for Fractions (DVD)
Video Observation Guide
Summary of Video Excerpts
Section 3: Choosing a Focus for Your Lesson Study Work
Path A: Introduce Fractions Using Linear Measurement Context
Flow of Japanese Lesson
Lesson Plans to Accompany DVD
Elementary School Teaching Guide for the Japanese Course of Study (Gr. 1-6)
Japanese Teaching Manuals: Translated Units 3B
Japanese Teaching Manuals: Translated Units 4B
Path B: Investigate An Aspect of Your Students' Fraction Number Sense
Investigation 1: Understanding that Fractions Are Accumulations of Unit Fractions
Investigation 2: Understanding the Relationship Between the Whole And a Fractional Part
Investigation 3: Understanding the Relative Size of Fractions
Investigation 4: Understanding Fractions on the Number Line
Investigation 5: Fraction Models as a Foundation for Understanding Calculation, Including Multiplication and Division of Fractions
Section 4: Planning, Conducting, and Discussing the Research Lesson
Teaching-Learning Plan for the Research Lesson (Template)
Protocol for Lesson Observation and Discussion
Lesson Observation Log
PowerPoint Presentation Template
End-of-Cycle Reflection Form
Section 5: Lesson Study Refresher: Overview and Suggestions for Getting Started
Setting Norms in Your Lesson Study Group
Choosing a Research Theme (Main Aim) for Lesson Study

Table 4: Teachers' Responses to Fraction Problems With Linear Measurement Context

	Highway Length	**Make $\frac{3}{4}$ Unit, Given $\frac{9}{8}$**
Correct	44%	65%
Incorrect	32%	21%
Did not attempt	24%	15%

Table 5: Student Responses to Fraction Problems With Linear Measurement Context

	% Students Answering Correctly - Pretest			
	Grade 2 (N=9)	Grade 3 (N=86)	Grade 4 (N=25)	Grade 6 (N=21)
Rabbit (integers)	11%	74%	56%	81%
Bear (fractions)	0%	27%	27%	57%
3 pieces of $\frac{[]}{4}$ inch is $\frac{3}{4}$	0%	22%	16%	90%
How many fourths make a whole?	(Not asked)	(Not asked)	60%	86%

Appendix A. Study of Fractions in Japanese Elementary Textbook (Hironaka & Sugiyama, 2006)

Grade level	Unit name	# of periods**	Instructional contents
3	Fractions	8	◎ Meaning of fractions and how to express them [fractions, denominator, numerator, $\frac{1}{10}$'s place] ○ Simple addition and subtraction of fractions
4	Fractions	11	◎ Concepts and structures of proper fractions, mixed numbers and improper fractions [proper fraction, mixed number, improper fraction] ○ How to show fractions on a number line, comparison of the size of fractions ○ Converting mixed numbers to improper fractions and vice-versa ◎ Relationship of equivalent fractions ◎ Addition and subtraction calculation of proper fractions with like denominators ○ Addition and subtraction calculation of mixed numbers with like denominators
5	Addition and Subtraction of fractions	12	◎ Meaning of simplifying a fraction and finding a common denominator and how to find them [simplifying a fraction, finding a common denominator] ◎ Addition and subtraction calculation of fractions with different denominators
5	Fractions and Decimal numbers	6	◎ The quotient of the division of whole numbers can be expressed as a fraction ○ Interrelationships between fractions, decimal numbers, and whole numbers
6	Multiplication and Division of Fractions	3	◎ Meaning of a fraction multiplied by a whole number and its method of calculation ◎ Meaning of a fraction divided by a whole number and its method of calculation
6	Multiplication of Fractions	11	◎ Meaning of multiplying by a fraction and its method of calculation ○ A product becomes smaller than multiplicand when the multiplicand is multiplied by a proper fraction (<1). ◎ Meaning of decimal calculation involving the second relationship of ratio (A = B x p). If B is the base quantity, A is the quantity to be compared, and p is the value of ratio. The quantity to be compared can be found by the base quantity multiplied by how many times as much as decimal number.) ○ In the case of fraction calculation, a calculation operation similar to whole numbers can be done
6	Division of Fractions	13	◎ Meaning of dividing by a fraction and its method of calculation ○ A quotient becomes larger than dividend when the dividend is divided by a proper fraction (<1). ○ Meaning of decimal calculation involving the first relationship of ratio (p = A ÷ B) and the third relationship of ratio (B= A ÷ p). ◎ Meaning of reciprocals and how to find them [reciprocals] ○ How to carry out mixed calculations of fractions and decimal numbers
6	*What Calculations Are We Going to Do?	(1)	○ Decision making on use of calculation operations, multiplication or division, involving fractions

◎	Indicates Important Content	*	Periods teachers can adjust by considering the students' state of learning
[ ]	Mathematical terms and symbols learned in the unit	( ) **	Number of available periods in the month 1 period is 45 minutes

Appendix B.

Lesson Plans to Accompany DVD: Introduction of Fractions in Linear Measurement Context (First Three Lessons in Unit)

Mathematics Lesson Plan for 3rd, 4th, and 5th grade

For the lessons on March 3, 4, 5, and 6
At the Mills College Children's School, Oakland, CA
Instructor: Akihiko Takahashi

a. Title of the Lesson: Fractions

b. Goals of the Unit:

Students will understand the meaning and the representations of fractions in simple cases and appropriately use them.

- To understand that fractions are used to express an amount obtained as a result of equal partitioning and are used to express quantities less than 1
- To understand that a fraction can be considered as a collection of unit fractions
- To understand fraction notation
- To become aware that a fraction can also be put on a number line like whole numbers
- To become aware that addition and subtraction can also be applied to fractions

c. Relationship of the Lesson to the Standards

Prior to this unit:

- Students understand the concepts of whole numbers that includes how to represent them and how to put them on a number line, and developed the ability to use numbers.
- Students become aware that fractions represent one portion of an equally divided object or a fractional part of some quantity from their everyday life.
- Students understand the concepts of length and capacity, and to measure them in simple cases.
  - Students know about the units to be used in measuring length (millimeter (*mm*), centimeter (*cm*), meter (*m*), and customary units such as mile).
  - Students know about the units to be used in measuring capacity (milliliter (*ml*), and liter (*l*).
- To understand the meaning of division and to use it.

↓

This Unit

↓

After this unit:

- Students will deepen their understanding of the meaning of fractions and be able compute fractions in simple cases.
- Students will deepen their understanding of the representation of fractions and their meanings. Furthermore, in simple cases to pay attention to the fact that there are equivalent fractions.
- Students will be able to add and subtract fractions with a common denominator.

d. Unit Plan

Day 1	Introduction to the unit	*Displaying the dates* Students will deepen their understanding of decimal notation through solving a problem related to children's everyday life.
Day 2	How can we express fractional parts (1) *Mathematics for elementary school 3B* (Hironaka H. et al., 2006) pp.57-58.	Students will become aware that fractions can be seen in students' everyday life. Students will understand that fractions are used to express an amount obtained as a result of equal partitioning and are used to express quantities less than 1 (only unit fractions).
Day 3	How can we express fractional parts (2) *Mathematics for elementary school 3B* (Hironaka H. et al., 2006) pp.58-59.	Students will understand that a fraction can be considered as a collection of unit fractions. Students will know fraction notation.
Day 4	The size of fraction *Mathematics for elementary school 3B* (Hironaka H. et al., 2006) pp.60	Student will become aware that a fraction can be put on a number line.

e. Instruction of the Lesson

Fraction is an important topic in the elementary grades. At the same time, it is one of the most challenging topics for students to understand (U.S. Department of Education, 2008).

Although many students have seen fractions in everyday life, e.g. a half mile on the highway road signs and a quarter pound in a fast-food-chain menu, these students may not be able to see fractions as numbers and use them comfortably like whole numbers. Researchers argue that the concept of fractions may be well introduced in second grade with manipulatives but they need to go through a gradual process moving from the concrete, the semi-concrete, and the abstract in order for them to see fractions as numbers (Gunderson & Gunderson, 1957).

According to the Japanese Course of Study Teaching Guide (Takahashi, Watanabe, & Yoshida, 2004), fractions should be introduced to represent one portion of an equally divided object, or to represent a fractional part of some quantity. After this kind of introduction, the idea that $\frac{2}{3}$ represents a collection of two $\frac{1}{3}$ units should be taught, as if $\frac{1}{3}$ is thought of as a unit (Thompson & Saldanha, 2003). Researchers also argue that students who work with fractions well use words in the beginning rather than the symbol, i.e., writing "2 thirds" rather than $\frac{2}{3}$ so that students can see "one third" as a unit just like measurement unit like miles (Gunderson & Gunderson, 1957). This shows that a fraction is the number that represents some portion of equally divided 1;

that is, a fraction has the meaning of $\frac{a}{b} = 1 \div b \times a$. (This means "*a* copies of $\frac{1}{b}$, since in Japan, the multiplicand is written first in multiplication expressions.)

It is also important for students to understand that fractions, just like whole numbers and decimal numbers, are used to represent not only size of numbers and quantities but also proportion of numbers and quantities. The Singapore National Curriculum (Ministry of Education, 2006) emphasizes that fractions are introduced as part of a whole interpretation in the primary 2 and the fraction of a set of objects, which is to represent proportion of numbers and quantities, should not be introduced until primary 4. Japanese Course of Study also mentions that part of a whole interpretation should be introduced at the beginning while using fractions to represent proportion of numbers and quantities should not be introduced until grade 5 (Takahashi, Watanabe, & Yoshida, 2008).

Based on the above discussion, the present research lesson unit, which consists of four lessons in four days, is designed for students to deepen their understanding of fractions in order for them to see that fractions are numbers. This unit is designed based on the English translation of the Japanese mathematics textbook series for the elementary grades, the most widely used public school mathematics textbook in Japan (Hironaka & Sugiyama, 2006). This series of research lessons support the following key ideas from the textbook:

- Fractions are introduced as part of whole interpretation, which is to express an amount obtained as a result of equal partitioning.
- Fractions are used to express quantities less than 1 in measurement contexts.
- Diagrams such as tape diagrams and area diagrams are used for students to understand that a fraction can be considered as a collection of unit fractions.
- Tape diagrams and number lines are used for students to see fractions are numbers just like whole numbers.

Since the Japanese textbooks are originally written in Japanese and designed for Japanese children who live in Japan, the followings are added to the contents of the Japanese textbook in order to maximize the benefits of learning for the English speaking children who live in the US.

- Some examples of fractions from students' everyday life will be shown in order to encourage students to see that fractions are often used in American society.
- Students will be given opportunities to write fractions using not only the symbol but also the word, i.e., writing "2 thirds" in addition to $\frac{2}{3}$ so that students can see "one third" as a unit.

f. Plan of the Lessons
Day 2
Goal of the lesson:
- Students become aware that fractions can be seen in students' everyday life.
- Students will understand that fractions are used to express an amount obtained as a result of equal partitioning and are used to express quantities less than 1 (only unit fractions).

Steps, Learning Activities Teacher's Questions and Expected Student Reactions	Teacher's Support	Points of Evaluation
1. Introduction Showing road signs with fractions to help students become aware that fractions can be found in everyday life. Ask students what they know about fractions • What does each number on the road sign represent? Students will discuss about what the fractions on the road sign represent by using their prior knowledge regarding whole numbers, measurements of distance, and fractions.	If some of the students seem not familiar with the term fraction, avoid using the term until the class informally defines the term. Encourage students to help each other to share their prior knowledge.	Is each student comfortable using term "fraction"? Does each student see what the fractions on the road sign represent?
2. Posing the Problem The length of the tape strip represents the length around trunk of a tree on the campus. The length of the tape strip is a bit longer than 1 *m*. How can we express the length of the fractional part of this tape strip using 1 *m* tape strip as a reference. **3. Anticipated Student Responses** • About a half meter • About a quarter meter • The length the fractional part is the same as the length of a portion that obtained by dividing 1 *m* into three equal parts. • One of the third of 1 *m* • $\frac{1}{3}$ • Some students might want to use their personal references such as the length of their hands or belongings. Some others may want to use yard or feet.	Each student will work with a partner. Each pair of students will use the actual length of the tape strips, one is 1 *m* and another is $1\frac{1}{3}$ *m.* The actual tape strips are similar to the diagrams on the textbook page. Encourage students to use 1 *m as* a reference to create their own unit in order to express the length of the fractional part.	Does each understand that the fractional part can be expressed by using 1 *m* as a reference?
4. Comparing and Discussing • To understand each approach to express the fractional part of the tape strip • To understand that a fraction can be used to express the length of a fractional part of the tape strip	Through the discussion encourage students to see that using a formal unit, such as meter, as a reference is a good idea to express quantities. Encourage students to write the length of the fractional part in the words, "1 third of 1 meter" or "1 third meter".	Does each student understand that the fractional part can be expressed using third meter as a unit

5. **Apply the learning to the similar situation** How to express the following parts of 1 *m* using the similar approach that you learned from the previous problem? • Find the length of a tape strip (1 half meter) • Find the length of a tape strip (1 fifth meter) Let's make 1 quarter-meter tape strip from 1 meter tape strip. • Ask a couple of younger grade students to explain how he/she made the tape strip • Ask other students to verify if the tape strips are 1 quarter-meter length.	Provide students actual length of tape strips. Encourage students to work with their partners. Once they find the length of each tape strip, let students write down the length in the words. Provide 1 meter tape strip for each student so that each of them can make own 1 quarter meter.	Does each student understand how to express the length and write it in the words? Does each student make a 1 quarter-meter length tape strip?
6. **Summing up** • Let each student write what he/she learned today.	Encourage students to use the board writing as an example in order to summarize what they learned.	

Evaluation:

- Do students understand that an amount obtained as a result of equal partitioning can be used to express quantities less than 1?
- Do students understand how to express the length of fractional parts by using words such as 1 third?

Picture of the road sign

Day 3

Goal of the lesson:

- Students will understand that a fraction can be considered as a collection of unit fractions.
- Students will know fraction notation.

Steps, Learning Activities Teacher's Questions and Expected Student Reactions	Teacher's Support	Points of Evaluation
1. Introduction Ask some students to share what they wrote in their notebook the day before.		
2. Posing the Problem The length of the tape strip is a bit shorter than 1 *m*. How can we express the length of this tape strip using 1 *m* tape strip as a reference. **3. Anticipated Student Responses** • A bit longer than a half meter • Twice as long as 1 third meter • Two of the third of 1 *m* • $\frac{2}{3}$ • 2 thirds meter.	Each student will work with a partner. Each pair of students will use the actual length of the tape strips, one is 1 *m* and another is $\frac{2}{3}$ *m.* The actual tape strips are similar to the diagrams in the textbook page but do not have dots line to show 1 third. Encourage students to use the tape strips from the day before lesson as reference.	Does each understand that the fractional part can be expressed by using 1 *m* as a reference?
4. Comparing and Discussing • To understand that the length of the fractional part is twice as long as the 1 third meter tape strip. • To understand that the fractional part can be expressed as a collection of third. • To understand the fractional part can be express by using the words, **2 thirds meter.**	Encourage students to write the length of the fractional part in the words, "2 thirds of 1 meter" or "2 thirds meter".	Does each student understand that the fractional part can be expressed using third meter as a unit?
5. **Apply the learning to another situation** How to express the following parts of 1 *liter* using the similar approach that you learned from the previous problem? • Find the amount of the water in the picture (2 fifths liter) • Find the amount of the water in the picture (1 fourth liter or 1 quarter liter) • Find the amount of the water in the picture (4 sixth liter)	Provide students a picture of the liter cup for each problem. Encourage students to work with their partners. Once they find the amount of the water, let students write down the amount in the words.	Does each student understand how to express the amount and write it in the words?
7. Summing up • Introduce fraction notation by replacing the words to express fractional parts.		

4 sixth liter can be written as $\frac{4}{6}$ liter. Numbers like $\frac{4}{6}$ are called **fractions.** **6 is called denominator to express the unit, sixth.** **4 is called numerator to express how many of the unit.** • Let students to use fraction notation to express the fractional part that they expressed in the words during the prior activities. • Do the exercises 2 & 3 on the textbook p.59. • Let each student write what he/she learned today.	Encourage students to use the board writing as an example in order to summarize what they learned.	Is each student able to express the quantities of the fractional parts by using fraction notation?

Evaluation:

- Do students understand that a fraction can be considered as a collection of unit fractions?
- Are students able to write fractions by using fraction notation?

Day 4

Goal of the lesson:

- Student will aware that a fraction can be put on a number line.

Steps, Learning Activities Teacher's Questions and Expected Student Reactions	Teacher's Support	Points of Evaluation
1. Introduction Ask some students to share what they wrote in their notebook the day before.		
2. Posing the Problem By using the tape strips we have had so far, lets create various length of the tape strips. Students can use the following unit fractions to express variety of fractions. · Half (meter) · Third (meter) · Quarter/fourth (meter) · Fifth (meter) · Sixth (liter/meter) **3. Anticipated Student Responses** • 2 halves meter • 2 thirds meter, 3 thirds meter • 2 fourths meter, 3 fourths meter, 4 fourths meter, 2 quarters meter, 3 quarters meter, 4 quarters meters • 2 fifths meters, 3 fifths meters, 4 fifths meters, 5 fifths meters • 2 sixths meters, 3 sixths meters, 4 sixths meters, 5 sixths meters, 6 sixths meters • $\frac{2}{2}, \frac{2}{3}, \frac{3}{3}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{6}{6}$	Each student will work with a partner. Each pair of students will use the tape strips from the previous days as a unit fraction in order to make several different lengths of tape strip. Each group will have a cash register paper role to make tape strips.	Does each understand that the fractional part can be expressed by using 1 *m* as a reference?
4. Comparing and Discussing • Through sharing the fractions, students will have opportunities to express, interpret, validate the fractions in the word and the fraction notation. • Organize fractions according to the size of the unit fraction, put the fractions with the same denominator on the same number line. • Compare the size of fractions with the same denominators.	Encourage students to write the length of the fractional part in the words, "2 thirds of 1 meter" or "2 thirds meter" and by the fraction notation.	Does each student understand that the fractional part can be expressed both in the words and fraction notation?
8. Summing up • Do the exercise in the textbook p.60. • Let each student write what he/she learned today.	Encourage students to use the board writing as an example in order to summarize what they learned.	

Evaluation:

- Are students able to put fractions on number lines?

Lesson Plan References:

Gunderson, A., & Gunderson, E. (1957). Fraction Concepts Held by Young Children. Arithmetic Teacher, 4 (October 1957), 168 - 173.

Hironaka, H., & Sugiyama, Y. (Eds.). (2006). Mathematics for Elementary School. Tokyo, Japan: Tokyo Shoseki Co., Ltd.

Ministry of Education, S. (2006). Mathematics Syllabus Primary.

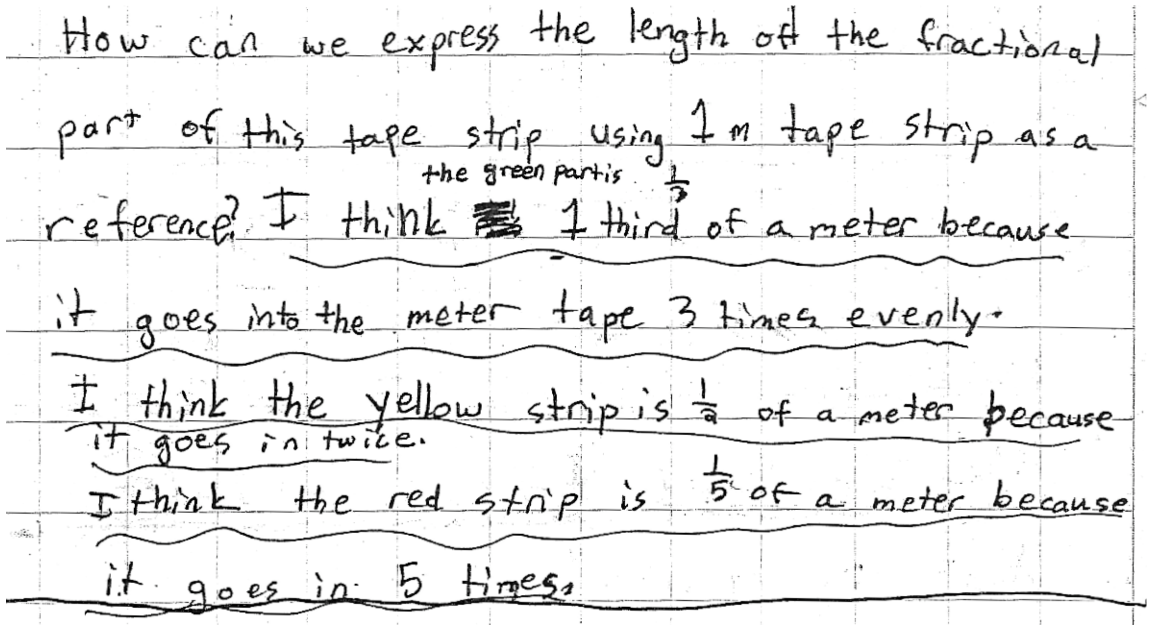
Takahashi, A., Watanabe, T., & Yoshida, M. (2004). Elementary School Teaching Guide for the Japanese Course of Study: Arithmetic (Grade 1-6). Madison, NJ: , Global Education Resources.

Takahashi, A., Watanabe, T., & Yoshida, M. (2008). English Translation of the Japanese Mathematics Curricula in the Course of Study, March, 2008, Grades 1-9. Madison, NJ: Global Education Resources.

Thompson, P. W., & Saldanha, L. i. A. (2003). Fractions and Multiplicative Reasoning. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), A Research Companion to Principles and Standards for School Mathematics (pp. 95-111): National Council of Teachers of Mathematics.

U.S. Department of Education (2008). Foundations for Success: The Final Report of the National Mathematics Advisory Panel. Washington, D.C.

Appendix C. Student Journal Entry from Field Study 1



How can we express the length off the fractional part of this tape strip using 1 m tape strip as a reference? I think the green part is $\frac{1}{3}$ 1 third of a meter because it goes into the meter tape 3 times evenly.

I think the yellow strip is $\frac{1}{2}$ of a meter because it goes in twice.

I think the red strip is $\frac{1}{5}$ of a meter because it goes in 5 times.

Appendix D: Photographs and Transcripts From Student Clinical Interview

The following photographs and transcript excerpts suggest this third-grade student understood that the length unit that fits into the whole *x* times is longer than the length unit that fits in *x* + 1 times.

INTERVIEWER: Can you find a third of that [drawn line] for me?
STUDENT: [See Figure a]

Figure a: Measuring $\frac{1}{3}$.

INTERVIEWER: What are you measuring?
STUDENT: Well, I'm mostly seeing what length could go in three times.
INTERVIEWER: So you're looking to make three times. OK. Why three?
STUDENT: Cause that's what… when it's three that would mean it's one third.
INTERVIEWER: OK, now if I was to show you that same line and ask you to find one fourth, could you find one fourth?
STUDENT: [see Figure b]

Figure b: To measure $\frac{1}{4}$, student shortens distance between finger and thumb.

INTERVIEWER: ….. OK, why did you put it in behind the other one? [see Figure c]

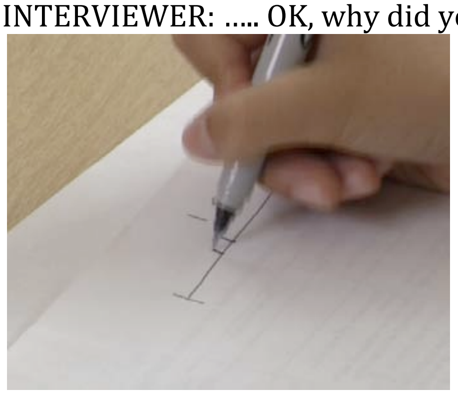


Figure c: Shows on number line how is a smaller unit than $\frac{1}{3}$.

STUDENT: Well, because the one fourth is smaller than the one third because it fits in more times.

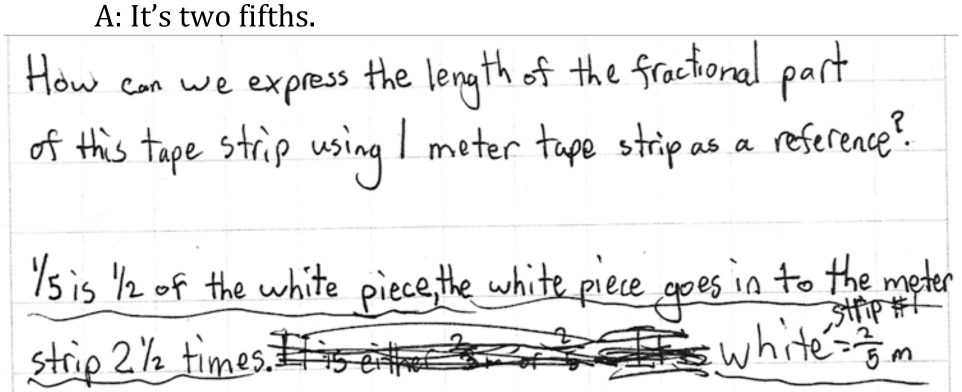
Appendix E: Students are asked to find the length of a "mystery strip" $\frac{2}{5}m$

Group A finds that the given "mystery strip" fits into a 1-*m* length $2\frac{1}{2}$ times, but cannot name the strip length. Other students call it a "two and a half" and: a) are confused by "half" in the name; b) think the length may be $\frac{2}{3}m$ because it includes three pieces (one full unit, one full unit, one half unit).

> L: OK, so this is one meter. So this is a part [shows a length between her hands equal to the length of the given strip], this is a section [shows a second length of the same size as the first], and then this is half of a section. …. I think it's two… But it couldn't be a third, because if it was a third it would fit in exactly. So we know it goes in two and a half times. So then how much would that be though?

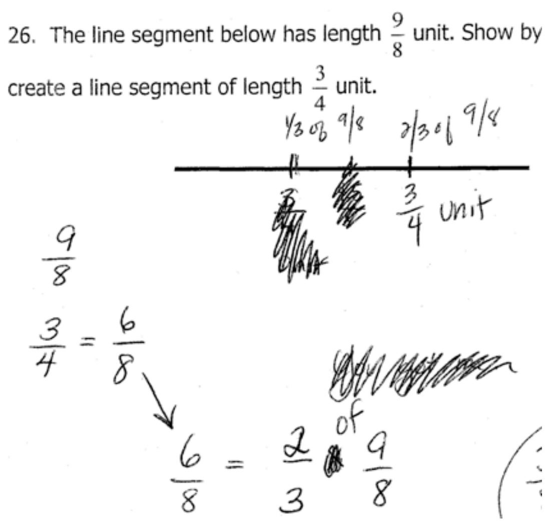
Group B identifies the unit fraction of $\frac{1}{5}m$, sees that the unit fraction repeats within the given strip two times, and knows that the given strip is $\frac{2}{5}m$.

> A: So, it's bigger, which means…
> I: Bigger than the one fourth and it's smaller than the…
> D: half…
> I: …and it's way bigger than the one fifth.
> A: Yeah.
> D: Wait, can I see it for a sec? [Student D takes strip length and walks to posted unit fraction lengths.] So…
> I: Measure the half.
> D: [Holds given strip length up to the half meter unit length.] No. [Seeing that it is shorter than the half.] …It's bigger than the green [one third] one.
> I: Try doing half of the one fifth.
> D: Yeah, that's good.
> A: It's two fifths.
> D: Yeah, it's half of the [points at red one fifth unit length].
> A: It's two fifths.

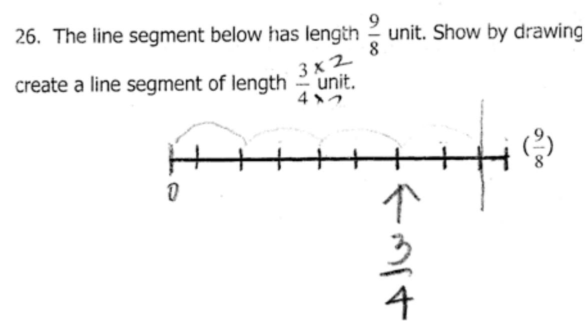


Appendix F: Strategies for creating a length $\frac{3}{4}$ unit from a length $\frac{9}{8}$ unit.

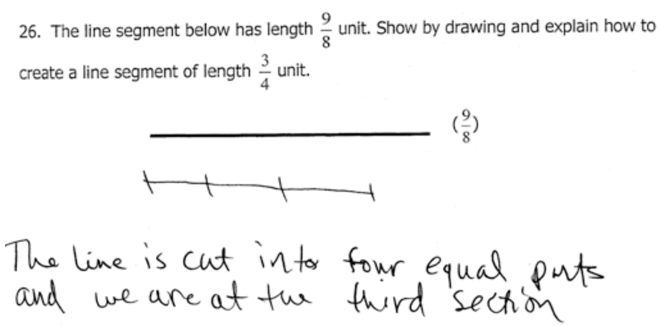
Correct Solution A – The teacher divided the $\frac{9}{8}$ whole into three separate $\frac{3}{8}$ pieces, knowing that $\frac{3}{4}$ = $\frac{6}{8}$ and that $\frac{6}{8}$ is $\frac{2}{3}$ of $\frac{9}{8}$.



Correct Solution B – The teacher identified each $\frac{1}{8}$ unit and the whole of $\frac{8}{8}$, and counted (shown with the arrows above the line) $3\frac{1}{4}$ lengths.



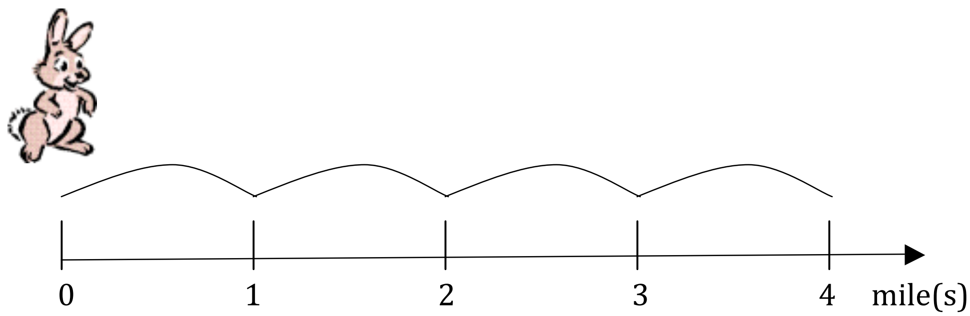
Incorrect Solution C – The teacher found $\frac{3}{4}$ of the length ($\frac{9}{8}$, which is $\frac{1}{8}$ more than 1 whole unit), rather than finding $\frac{3}{4}$ of 1 whole unit.



Appendix G. Selected Student Assessment Items

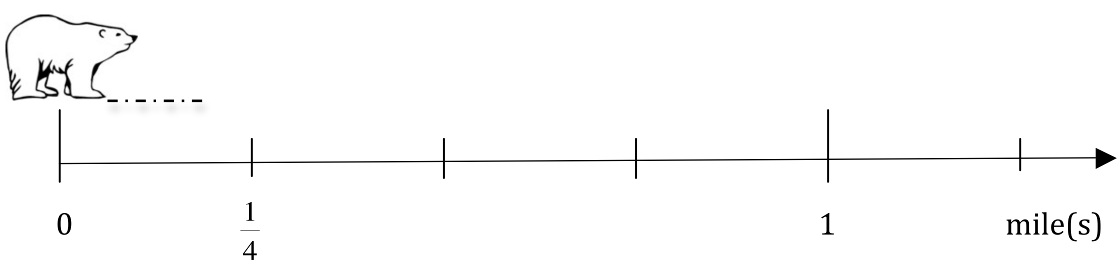
A rabbit travels 1 mile every day. If the rabbit starts at 0, fill in the blank and draw an X on the number line below to show where the rabbit will be after 3 days of travel. (Adapted from Saxe, 2009.)

Answer: _________ mile(s)



A bear travels $\frac{1}{4}$ mile in one day. If the bear starts at 0, fill in the blank and draw an X on the number line below to show where the bear will be after 3 days of travel. (Adapted from Saxe, 2009.)

Answer: _________ mile(s)



Fill in the empty boxes with the missing numbers in the problem below:
(Adapted from Japanese teachers' manual, supplementary volume.)

3 pieces of $\frac{1}{4}$ inch is $\frac{\square}{\square}$ inch

How many fourths make a whole? Answer: _____________
(Adapted from Japanese teachers' manual, supplementary volume.)